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Quantum KZ equation with $|q|= 1$ and correlation functions of the XXZ model in the gapless regime

Michio Jimbo† and Tetsuji Miwa‡

† Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan
‡ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

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Abstract. An integral solution to the quantum Knizhnik–Zamolodchikov (qKZ) equation with $|q| = 1$ is presented. Upon specialization, it leads to a conjectural formula for correlation functions of the XXZ model in the gapless regime. The validity of this conjecture is verified in special cases, including the nearest-neighbour correlator with an arbitrary coupling constant and general correlators in the XXX and XY limits.

1. Introduction

Consider the one-dimensional spin-$\frac{1}{2}$ XXZ chain

$$H = -\tfrac{1}{2} \sum_{n=-\infty}^{\infty} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z\right). \tag{1.1}$$

In this paper we address the problem of describing correlation functions of (1.1) in the gapless regime $|\Delta| \leqslant 1$. In earlier works [1, 2], the case of the anti-ferromagnetic regime $\Delta < -1$ was treated within the framework of the representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$. As a result, correlation functions have been described by using the quantum Knizhnik–Zamolodchikov (qKZ) equation. It is with this aspect that we will be concerned in this paper. Before coming to the main subject of this paper, let us first recall some known results for $\Delta < -1$.

Let $V = \mathbb{C}^2$, and consider the R-matrix $R(\beta) \in \mathrm{End}_{\mathbb{C}}(V \otimes V)$ associated with the XXZ model (see equation (2.2)). The qKZ equation is the following system of linear difference equations for an unknown function $G_n(\beta_1, \ldots, \beta_{2n})$ that takes values in $V^{\otimes 2n}$:

$$\begin{aligned} G_n(\beta_1, \ldots, \beta_j - 2\pi\mathrm{i}, \ldots, \beta_{2n}) &= R_{j\, j+1}(\beta_j - \beta_{j+1} - 2\pi\mathrm{i})^{-1} \cdots R_{j\, 2n}(\beta_j - \beta_{2n} - 2\pi\mathrm{i})^{-1} \\ &\quad \times R_{1\, j}(\beta_1 - \beta_j) \cdots R_{j-1\, j}(\beta_{j-1} - \beta_j) G_n(\beta_1, \ldots, \beta_j, \ldots, \beta_{2n}). \end{aligned} \tag{1.2}$$

Here $R_{ij}(\beta) \in \mathrm{End}_{\mathbb{C}}(V^{\otimes 2n})$ signifies the matrix acting as $R(\beta)$ on the (i, j)th tensor components and as identity elsewhere. The correlation functions of arbitrary local operators are obtained as the specialization

$$G_n(\overbrace{\beta + \pi\mathrm{i}, \ldots, \beta + \pi\mathrm{i}}^{n}, \overbrace{\beta, \ldots, \beta}^{n}). \tag{1.3}$$

To be precise, in the case $\Delta < -1$, there are two functions $F_n^{(i)}$ $(i = 0, 1)$ associated with the two anti-ferromagnetic vacuum states, and it is their sum $G_n = F_n^{(0)} + F_n^{(1)}$ that satisfies the qKZ equation (1.2), as well as a set of relations (2.4)–(2.6). The correlators are given by specializations of $F_n^{(i)}$ rather than G_n itself. In the context of representation theory, the functions $F_n^{(i)}$ are traces of products of certain intertwiners (vertex operators) taken over the integrable highest weight modules $V(\Lambda_i)$. By realizing $V(\Lambda_i)$ in terms of bosonic free fields, an explicit integral formula was obtained for the functions $F_n^{(i)}$ $(i = 0, 1)$ and hence for the solution G_n of the qKZ equation.

The argument relating correlation functions to the functions G_n is based on the extension of the corner transfer matrix method [2, 3]. It is applicable to the more general case of the XYZ spin chain as well [4]. Correlation functions are related in the same way as above with solutions G_n of the qKZ equation, this time having the elliptic R-matrix as coefficients. Unfortunately, the mathematical structure of the XYZ model is not yet fully understood (see [5, 6] for a formulation of an elliptic extension of $U_q(\widehat{sl}_2)$). The free-field realization is still unavailable (see, however, the recent development [7, 8] in this direction). Thus it remains an important open problem to construct solutions to the qKZ equation in the elliptic case.

For the XXZ chain in the gapless regime $|\Delta| \leqslant 1$, the corner transfer matrix fails to be well defined. Nevertheless, this case can be viewed as a limiting case of the XYZ chain, so that the same recipe (1.3) is expected to apply for obtaining correlation functions. (Unlike the case $\Delta < -1$, the vacuum state is unique and the distinction between $F^{(0)}$ and $F^{(1)}$ disappears.) The problem is then to find appropriate solutions of the qKZ equation.

Up to an overall scalar, the R-matrix $R(\beta)$ of the XXZ chain is a rational function in

$$\zeta = \mathrm{e}^{-\nu\beta} \qquad q = -\mathrm{e}^{\pi\mathrm{i}\nu} \tag{1.4}$$

where

$$\Delta = -\cos\pi\nu = \frac{q + q^{-1}}{2}\,. \tag{1.5}$$

However, the nature of the solutions is quite different depending on whether $\Delta < -1$ or $|\Delta| \leqslant 1$. In the case $\Delta < -1$, we have $-1 < q < 0$ and the solutions are meromorphic in ζ, typically involving infinite products of the form $\prod_{n=1}^{\infty}(1 - \zeta q^{2n})$. On the other hand, the case $|\Delta| \leqslant 1$ corresponds to $|q| = 1$. There are no analytic solutions which are single-valued in ζ. Instead one has to look for solutions which are meromorphic in $\log\zeta$.

A certain class of solutions to the qKZ equation with $|q| = 1$ has been studied in detail by Smirnov [9] in connection with the form factors in the sine–Gordon theory. The equation relevant to the correlation functions of the XXZ model is slightly different from Smirnov's, in particular the shift $-2\pi\mathrm{i}$ in (1.2) is replaced by $+2\pi\mathrm{i}$ in his case (the former has 'level -4' while the latter has 'level 0'; we will also consider the case in which the shift $-2\pi\mathrm{i}$ is replaced by $-\mathrm{i}\lambda$ where $\lambda > 0$ is a parameter.) In this paper we give a solution to the former, in the form of an integral which has a similar structure to the case $\Delta < -1$, and conjecture that its specialization (1.3) gives the correlation functions of the XXZ model with $|\Delta| \leqslant 1$. Our integral formula is essentially the same as the one written down earlier by Lukyanov [10]. However, in Lukyanov's case, it is given as a generating function of the form factors of local operators in the sine–Gordon theory. Our point here is to interpret it as a formula for the correlation functions *on the lattice*. In general, difference equations determine the solutions only up to arbitrary periodic functions. We need to ensure that the particular solution we present actually corresponds to correlation functions. As supporting evidence, we verify this statement in three special cases for which exact results are available: (i) the nearest-neighbour correlation $\langle\sigma_1^z\sigma_2^z\rangle$, (ii) the XXX model $\Delta = -1$ and (iii) the XY model $\Delta = 0$. The integral formula and these verifications are the main results of this paper.

The text is organized as follows. In section 2 we formulate the qKZ and related equations. We then write down an integral formula for solutions. In section 3 we specialize the formula in section 2 and propose that it gives correlation functions. In the special case $\nu = 0$, we recover the formula for the correlation functions of the XXX model derived earlier in [2, 11, 12]. In section 4 we process our integral formula to reproduce the simplest correlation function $\langle\sigma_1^z\sigma_2^z\rangle$. This quantity can be derived by differentiating the ground-state energy of the Hamiltonian. In section 5 we consider the XY limit ($\nu = \frac{1}{2}$), which can be studied independently by using free fermions. The correlation functions are given by the determinants of certain matrices whose entries are elementary functions in β_j. Our integral formula in this case is shown to be equivalent to the free-fermion result. Section 6 is devoted to a discussion concerning some previous works [9, 10, 12] on the qKZ equation with $|q| = 1$.

Since most of the statements are proved by purely computational means, we have put technical points in the appendices in order to make the paper easier to read. In appendix A, a summary of Barnes' multiple gamma functions is offered. Appendix B contains the proof of the difference equations of section 2. In appendix C it is shown how the n-fold integral is reduced to an $(n-1)$-fold one by explicitly carrying out the integration once. Appendix D is the derivation of the expression for $\langle\sigma_1^z\sigma_2^z\rangle$. Appendix E is the evaluation of an integral in the case $\nu = \frac{1}{2}$. Finally, appendix F is devoted to the free-fermion theory in the XY limit.

2. Integral formula

2.1. The difference equations

In this section, we formulate the system of equations we are going to study, including the qKZ equation with $|q| = 1$. We then give a particular solution in the form of an integral. Throughout this section we fix parameters ν and λ (see equation (1.4)) such that $0 < \nu < 1$ and $\lambda > 0$. For the convergence of the integral, we assume that

$$\lambda + \frac{\pi}{\nu} > 2\pi \,. \tag{2.1}$$

In the application to the XXZ model, we will choose $\lambda = 2\pi$.

Consider the R-matrix $R(\beta) \in \mathrm{End}_{\mathbb{C}}(V \otimes V)$ acting on the tensor product of $V = \mathbb{C}v^+ \oplus \mathbb{C}v^-$:

$$\begin{aligned} R(\beta)(v^{\varepsilon_1'} \otimes v^{\varepsilon_2'}) &= \sum_{\varepsilon_1,\varepsilon_2} R_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}(\beta) v^{\varepsilon_1} \otimes v^{\varepsilon_2} \\ R(\beta) &= \frac{1}{\kappa(\beta)} \bar{R}(\beta)\,. \end{aligned} \tag{2.2}$$

The parameter ν enters the matrix elements as follows:

$$\begin{aligned} &\bar{R}_{++}^{++}(\beta) = \bar{R}_{--}^{--}(\beta) = 1 \\ &\bar{R}_{+-}^{+-}(\beta) = \bar{R}_{-+}^{-+}(\beta) = \bar{b}(\beta) \\ &\bar{R}_{-+}^{+-}(\beta) = \bar{R}_{+-}^{-+}(\beta) = \bar{c}(\beta) \\ &\bar{R}_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}(\beta) = 0 \qquad \text{in the other cases} \end{aligned} \tag{2.3}$$

where

$$\bar{b}(\beta) = \frac{\sinh \nu\beta}{\sinh \nu(\pi\mathrm{i} - \beta)} \qquad \bar{c}(\beta) = \frac{\sinh \nu\pi\mathrm{i}}{\sinh \nu(\pi\mathrm{i} - \beta)}\,.$$

The function $\kappa(\beta)$ will be specified below. It is chosen to ensure that the R-matrix satisfies the unitarity and the crossing symmetry relations

$$R_{12}(\beta)R_{21}(-\beta) = \mathrm{id} \qquad R^{\varepsilon_2'\varepsilon_1}_{\varepsilon_2\varepsilon_1'}(\beta) = R^{-\varepsilon_1'\,\varepsilon_2'}_{-\varepsilon_1\,\varepsilon_2}(\pi\mathrm{i}-\beta)\,.$$

Let n be a non-negative integer. Consider a $V^{\otimes 2n}$-valued function $G_n = G_n(\beta_1,\ldots,\beta_{2n})$, depending on the 'spectral parameters' $\beta_1,\ldots,\beta_{2n}$. We set

$$G_0 = 1\,.$$

We study the following system of difference equations for G_n involving the parameter λ:

$$G_n(\ldots,\beta_{j+1},\beta_j,\ldots)_{\ldots,\varepsilon_{j+1},\varepsilon_j,\ldots} = \sum_{\varepsilon_j',\varepsilon_{j+1}'} R^{\varepsilon_j',\varepsilon_{j+1}'}_{\varepsilon_j,\varepsilon_{j+1}}(\beta_j-\beta_{j+1})G_n(\ldots,\beta_j,\beta_{j+1},\ldots)_{\ldots,\varepsilon_j',\varepsilon_{j+1}',\ldots} \tag{2.4}$$

$$G_n(\beta_1,\ldots,\beta_{2n-1},\beta_{2n}-\mathrm{i}\lambda)_{\varepsilon_1,\ldots,\varepsilon_{2n}} = G_n(\beta_{2n},\beta_1,\ldots,\beta_{2n-1})_{\varepsilon_{2n},\varepsilon_1,\ldots,\varepsilon_{2n-1}} \tag{2.5}$$

$$G_n(\beta_1,\ldots,\beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{2n}}\Big|_{\beta_{2n}=\beta_{2n-1}+\pi\mathrm{i}} = \delta_{\varepsilon_{2n-1}+\varepsilon_{2n},0}G_{n-1}(\beta_1,\ldots,\beta_{2n-2})_{\varepsilon_1,\ldots,\varepsilon_{2n-2}}\,. \tag{2.6}$$

In particular, the qKZ equation (1.2) is a consequence of (2.4) and (2.5). It can be shown also that (2.4) and (2.6) imply

$$\begin{aligned} &G_n(\beta_1,\ldots,\beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{2n}}\Big|_{\beta_{j+1}=\beta_j+\pi\mathrm{i}} \\ &\quad = \delta_{\varepsilon_j+\varepsilon_{j+1},0}G_{n-1}(\beta_1,\ldots,\beta_{j-1},\beta_{j+2},\ldots,\beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{j-1},\varepsilon_{j+2},\ldots,\varepsilon_{2n}} \end{aligned} \tag{2.7}$$

for any $j = 1,\ldots,2n-1$. Note that equations (2.4)–(2.6) involve only the functions $G_n(\beta_1,\ldots,\beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{2n}}$ with fixed value of the 'spin' $\varepsilon_1+\cdots+\varepsilon_{2n}$. Throughout this paper we will restrict ourselves to the 'spin-0' case, i.e. we assume

$$G_n(\beta_1,\ldots,\beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{2n}} = 0 \qquad \text{unless} \quad \varepsilon_1+\cdots+\varepsilon_{2n} = 0\,.$$

2.2. Auxiliary functions

Our aim in this section is to construct a solution to (2.4)–(2.6) by using an n-fold integral. The formula involves certain special functions $\kappa(\beta), \rho(\beta), \varphi(\beta), \psi(\beta)$. Let us first give their definitions and list some of their properties. In what follows, $S_r(x|\omega_1,\ldots,\omega_r)$ will denote the multiple sine function (see appendix A for the definition).

- $\kappa(\beta)$

$$\begin{aligned} \kappa(\beta) &= -\frac{S_2(\mathrm{i}\beta|2\pi,\pi/\nu)S_2(\pi-\mathrm{i}\beta|2\pi,\pi/\nu)}{S_2(-\mathrm{i}\beta|2\pi,\pi/\nu)S_2(\pi+\mathrm{i}\beta|2\pi,\pi/\nu)} \\ &= \exp\left\{-\mathrm{i}\int_0^\infty \frac{\mathrm{d}t}{t}\,\frac{\sin(2\beta\nu/\pi)t\,\sinh(1-\nu)t}{\sinh t\,\cosh\nu t}\right\} \\ \kappa(\beta)&\kappa(-\beta) = 1 \\ \kappa(\beta)&\kappa(\beta-\pi\mathrm{i}) = \frac{\sinh\nu\beta}{\sinh\nu(\pi\mathrm{i}-\beta)} = \bar{b}(\beta)\,. \end{aligned}$$

- $\rho(\beta)$

$$\begin{aligned} \rho(\beta) &= \sinh\frac{\pi\beta}{\lambda}\,\frac{S_3(\pi-\mathrm{i}\beta)S_3(\pi+\lambda+\mathrm{i}\beta)}{S_3(-\mathrm{i}\beta)S_3(\lambda+\mathrm{i}\beta)} \qquad (S_3(x) = S_3(x|2\pi,\lambda,\pi/\nu)) \\ &= \rho\left(\frac{\mathrm{i}\lambda}{2}\right)\exp\left\{\int_0^\infty \frac{\mathrm{d}t}{t}\,\frac{\sin^2((\beta-\mathrm{i}\lambda/2)\nu t/\pi)\,\sinh(1-\nu)t}{\cosh\nu t\ \sinh t\ \sinh\lambda\nu t/\pi}\right\} \end{aligned}$$

$$\frac{\rho(\beta)}{\rho(-\beta)} = \kappa(\beta) \tag{2.8}$$

$$\rho(\mathrm{i}\lambda - \beta) = \rho(\beta) \tag{2.9}$$

$$\rho(\beta) = \frac{\nu\rho(\pi\mathrm{i})}{\mathrm{i}\sin\pi\nu}(\beta + \pi\mathrm{i}) + \cdots \quad \text{when } \beta \to -\pi\mathrm{i}\,. \tag{2.10}$$

- $\varphi(\beta)$

$$\varphi(\beta) = \frac{2}{S_2(\pi/2 + \mathrm{i}\beta|\lambda, \pi/\nu)S_2(\pi/2 - \mathrm{i}\beta|\lambda, \pi/\nu)}$$

$$= \varphi(0)\exp\left\{-2\int_0^\infty \frac{\mathrm{d}t}{t}\,\frac{\sin^2\beta\nu t/\pi\,\sinh(1 + (\lambda - \pi)\nu/\pi)t}{\sinh\lambda\nu t/\pi\,\sinh t}\right\}$$

$$\varphi(-\beta) = \varphi(\beta)$$

$$\frac{\varphi(\beta - \mathrm{i}\lambda)}{\varphi(\beta)} = -\frac{\sinh\nu(\beta - \pi\mathrm{i}/2)}{\sinh\nu(\beta + \pi\mathrm{i}/2 - \mathrm{i}\lambda)} \tag{2.11}$$

$$\frac{\varphi(\beta \pm \pi\mathrm{i}/\nu)}{\varphi(\beta)} = -\frac{\sinh(\pi/\lambda)(\beta \pm \pi\mathrm{i}/2)}{\sinh(\pi/\lambda)(\beta \mp \pi\mathrm{i}/2 \pm \frac{\pi\mathrm{i}}{\nu})}$$

$$\varphi(\beta) = \pm\frac{\sqrt{\lambda/(\pi\nu)}}{\mathrm{i}S_2(\pi|\lambda, \pi/\nu)(\beta \mp \pi\mathrm{i}/2)} + \cdots \qquad \text{when } \beta \to \pm\pi\mathrm{i}/2 \tag{2.12}$$

$$\rho(\beta)\rho(\beta + \pi\mathrm{i})\varphi(\beta + \pi\mathrm{i}/2) = \frac{\mathrm{i}}{4\sinh\nu\beta}\,. \tag{2.13}$$

- $\psi(\beta)$

$$\psi(\beta) = \sinh\pi\beta/\lambda\; S_2(\pi + \mathrm{i}\beta|\lambda, \pi/\nu)S_2(\pi - \mathrm{i}\beta|\lambda, \pi/\nu)$$

$$\psi(-\beta) = -\psi(\beta) \tag{2.14}$$

$$\varphi(\beta + \pi\mathrm{i}/2)\varphi(\beta - \pi\mathrm{i}/2)\psi(\beta) = \frac{1}{\sinh\nu\beta} \tag{2.15}$$

$$\frac{\psi(\beta + \mathrm{i}\lambda)}{\psi(\beta)} = \frac{\sinh\nu(\beta + \mathrm{i}\lambda - \pi\mathrm{i})}{\sinh\nu(\beta + \pi\mathrm{i})}\,. \tag{2.16}$$

In addition we will use a constant c_n given by

$$c_n = (-16)^{n(n-1)/2}\left(\frac{\pi S_2(\pi|\lambda, \pi/\nu)^2}{\lambda\rho(\pi\mathrm{i})}\right)^n. \tag{2.17}$$

In the case $\lambda = 2\pi$ the formulae simplify to

$$\psi(\beta) = \sinh\beta \qquad c_n = \frac{(-16)^{n(n-1)/2}}{\rho(\pi\mathrm{i})^n}\,.$$

2.3. Integral formula

Let us present the integral formula for $G_n(\beta_1, \ldots, \beta_{2n})_{\varepsilon_1\ldots\varepsilon_{2n}}$. Given a set of indices $\varepsilon_1, \ldots, \varepsilon_{2n} \in \{+, -\}$, we define a map $a \in \{1, \ldots, n\} \to \bar{a} \in \{1, \ldots, 2n\}$ in such a way that $\varepsilon_{\bar{a}} = +$ and $\bar{a} < \bar{b}$ if $a < b$. Define further a meromorphic function

$$Q_n(\alpha|\beta)_{\varepsilon_1\ldots\varepsilon_{2n}} = \frac{\prod_{j<\bar{a}}\sinh\nu(\alpha_a - \beta_j + \pi\mathrm{i}/2)\prod_{j>\bar{a}}\sinh\nu(\beta_j - \alpha_a + \pi\mathrm{i}/2)}{\prod_{a<b}\sinh\nu(\alpha_a - \alpha_b - \pi\mathrm{i})}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_{2n})$. After these preparations, we set

$$G_n(\beta_1, \ldots, \beta_{2n})_{\varepsilon_1 \ldots \varepsilon_{2n}} = c_n \prod_{j<k} \rho(\beta_j - \beta_k) \times \prod_a \int_{C_a} \frac{d\alpha_a}{2\pi i} \prod_{a,j} \varphi(\alpha_a - \beta_j) \prod_{a<b} \psi(\alpha_a - \alpha_b) Q_n(\alpha|\beta)_{\varepsilon_1 \ldots \varepsilon_{2n}} . \tag{2.18}$$

Clearly we have, for any γ,

$$G_n(\beta_1 + \gamma, \ldots, \beta_{2n} + \gamma) = G_n(\beta_1, \ldots, \beta_{2n}) .$$

In appendix B, we will prove that with the appropriate choice of the integration contours C_a $(1 \leqslant a \leqslant n)$ as given below, the function $G_n(\beta_1, \ldots, \beta_{2n})$ is meromorphic and satisfies (2.4)–(2.6).

In order to specify the integration contours, let us examine the poles of the integrand of (2.18). The poles of $\varphi(\alpha_a - \beta_j)$ are at

$$\alpha_a - \beta_j = \pm i(n_1\lambda + n_2/\nu\pi + \pi/2) \qquad (n_1, n_2 \geqslant 0) . \tag{2.19}$$

The poles of $\psi(\alpha_a - \alpha_b)$ $(a < b)$ are at

$$\alpha_a - \alpha_b = \pm i(n_1\lambda + (n_2 - \nu)\pi/\nu) \qquad (n_1, n_2 \geqslant 1) \tag{2.20}$$

and the poles of $1/\sinh \nu(\alpha_a - \alpha_b - \pi i)$ are at

$$\alpha_a - \alpha_b = \frac{n + \nu}{\nu}\pi i \qquad (n \in \mathbb{Z}) . \tag{2.21}$$

Since $\psi(\alpha_a - \alpha_b)$ has zeros at

$$\alpha_a - \alpha_b = \pm i(n_1\lambda + n_2\pi/\nu + \pi) \qquad (n_1, n_2 \geqslant 0) \tag{2.22}$$

the poles of $\psi(\alpha_a - \alpha_b)/\sinh \nu(\alpha_a - \alpha_b - \pi i)$ are at

$$\alpha_a - \alpha_b = i(n_1\lambda + \frac{n_2 - \nu}{\nu}\pi) \qquad (n_1, n_2 \geqslant 1) \tag{2.23}$$

or

$$\alpha_a - \alpha_b = -i(n_1\lambda + \frac{n_2 - \nu}{\nu}\pi) \qquad (n_1 \geqslant 0, n_2 \geqslant 1) . \tag{2.24}$$

Therefore, the poles in the variable α_a of the integrand are contained in the set

$$\begin{aligned}
&\{\beta_j \pm i(n_1\lambda + \frac{n_2}{\nu}\pi + \frac{\pi}{2}) && (1 \leqslant j \leqslant 2n; n_1, n_2 \geqslant 0) \\
&\alpha_b + i(n_1\lambda + \frac{n_2 - \nu}{\nu}\pi) && (a < b \leqslant n; n_1, n_2 \geqslant 1) \\
&\alpha_b - i(n_1\lambda + \frac{n_2 - \nu}{\nu}\pi) && (a < b \leqslant n; n_1 \geqslant 0, n_2 \geqslant 1) \\
&\alpha_b + i(n_1\lambda + \frac{n_2 - \nu}{\nu}\pi) && (1 \leqslant b < a; n_1 \geqslant 0, n_2 \geqslant 1) \\
&\alpha_b - i(n_1\lambda + \frac{n_2 - \nu}{\nu}\pi) && (1 \leqslant b < a; n_1, n_2 \geqslant 1)\} .
\end{aligned}$$

We choose the contour C_a for α_a by the following rule:

$$\alpha_a \text{ lies on the real line for } |\alpha_a| \gg 0 \tag{2.25}$$

$$\begin{aligned}
&\beta_j + \tfrac{\pi i}{2} \quad (1 \leqslant j \leqslant 2n) \qquad \alpha_b + i(\lambda + \tfrac{1-\nu}{\nu}\pi) \quad (a < b \leqslant n) \\
&\qquad \alpha_b + i\tfrac{1-\nu}{\nu}\pi \quad (1 \leqslant b < a) \qquad \text{are above } C_a
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
&\beta_j - \tfrac{\pi i}{2} \quad (1 \leqslant j \leqslant 2n) \qquad \alpha_b - i\tfrac{1-\nu}{\nu}\pi \quad (a < b \leqslant n) \\
&\qquad \alpha_b - i(\lambda + \tfrac{1-\nu}{\nu}\pi) \quad (1 \leqslant b < a) \qquad \text{are below } C_a .
\end{aligned} \tag{2.27}$$

Note that we can choose C_a to be the same contour C for all a such that $\beta_j + \pi i/2$ $(1 \leqslant j \leqslant 2n)$ are above C and $\beta_j - \pi i/2$ $(1 \leqslant j \leqslant 2n)$ are below C.

Let us check the convergence of the integral. Recall that the periods of the double sine function S_2 used in φ and ψ are such that $\omega_1 = \lambda > 0$ and $\omega_2 = \pi/\nu > \pi$. In the proof below, we use 'constant' to mean different constants which appear in the estimates. From (A.14), we have

$$|\varphi(\alpha_a - \beta_j)| \leqslant \text{constant } \times \mathrm{e}^{\frac{\pi(\pi-\omega_1-\omega_2)}{\omega_1\omega_2}|\alpha_a|}$$

$$\left|\frac{\psi(\alpha_a - \alpha_b)}{\sinh\nu(\alpha_a - \alpha_b - \pi\mathrm{i})}\right| \leqslant \text{constant } \times \mathrm{e}^{\frac{2\pi(\omega_2-\pi)}{\omega_1\omega_2}(|\alpha_a|+|\alpha_b|)}$$

$$|\sinh\nu(\alpha_a - \beta_j \pm \pi\mathrm{i}/2)| \leqslant \text{constant } \times \mathrm{e}^{\frac{\pi}{\omega_2}|\alpha_a|}.$$

Collecting these estimates we see that

$$\left|\prod_{a,j}\varphi(\alpha_a - \beta_j)\prod_{a<b}\psi(\alpha_a - \alpha_b)Q_n(\alpha|\beta)_{\varepsilon_1\ldots\varepsilon_{2n}}\right| \leqslant \text{constant } \times \mathrm{e}^{\frac{\pi(2\pi-\omega_1-\omega_2)}{\omega_1\omega_2}\sum_a|\alpha_a|}.$$

Since we have assumed that $2\pi - \omega_1 - \omega_2 = 2\pi - \lambda - \pi/\nu < 0$, the integral is convergent.

2.4. One-time integration

In the case of interest $\lambda = 2\pi$, the n-fold integral for $G_n(\beta_1, \ldots, \beta_{2n})$ can be reduced to an $(n-1)$-fold integral by carrying out the integration once. The result is stated as follows.

$$\begin{aligned} G_n(\beta_1, \ldots, \beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{2n}} &= \tilde{c}_n \prod_{1\leqslant j<k\leqslant 2n}\rho(\beta_j - \beta_k) \\ &\times \frac{\pi}{\nu}\frac{1}{\mathrm{e}^{\sum\beta_j/2}\sum \mathrm{e}^{-\beta_j}}\sum_{l=1}^{n}(-1)^{\bar{l}+1}\int\ldots\int\prod_{k\neq l}\mathrm{d}\alpha_k \mathrm{D}(\alpha_1, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_n) \\ &\times \prod_{k\neq l}\left[\prod_{j=1}^{2n}\varphi(\alpha_k - \beta_j)\prod_{j<\bar{k}}\sinh\nu\big(\alpha_k - \beta_j + \pi\mathrm{i}/2\big)\prod_{j>\bar{k}}\sinh\nu\big(-\alpha_k + \beta_j + \pi\mathrm{i}/2\big)\right] \\ &\times \frac{\sinh\nu\big(\sum_{k\neq l}\alpha_k + \frac{1}{2}\beta_{\bar{l}} - \frac{1}{2}\sum_{j\neq\bar{l}}\beta_j + \pi\mathrm{i}(\bar{l} - 2l + \frac{1}{2})\big)}{\prod_{r<s,r,s\neq l}\sinh\nu(\alpha_r - \alpha_s - \pi\mathrm{i})}. \end{aligned} \tag{2.28}$$

Here we have set

$$\tilde{c}_n = 2^{3n(n-1)/2}\big(-\pi\mathrm{i}\rho(\pi\mathrm{i})\big)^{-n}$$

and

$$\mathrm{D}(x_1, \ldots, x_{n-1}) = \det\left(\mathrm{e}^{-(n-2k-1)x_j}\right)_{1\leqslant j,k\leqslant n-1}.$$

As before, the numbers $\bar{1} < \cdots < \bar{n}$ are determined by

$$\{\bar{l} \mid 1 \leqslant l \leqslant n\} = \{j \mid 1 \leqslant j \leqslant 2n, \varepsilon_j = +\}.$$

The integration is taken along a path going from $-\infty$ to $+\infty$ in such a way that $-\pi/2 < \mathrm{Im}(\alpha_k - \beta_j) < \pi/2$ for all k, j. In the above, we assume that $0 < \nu < \frac{1}{2}$ for the convergence of the integral. It should also be possible to treat the case $\frac{1}{2} \leqslant \nu < 1$ by introducing a suitable regularization as in [9], but we do not go into this question here.

The derivation of (2.28) will be given in appendix C.

3. Correlation functions

We now proceed to the description of correlation functions of the XXZ model. From now on, we assume that $\lambda = 2\pi$.

First let us set up the notation. Let $E_{\varepsilon\varepsilon'}$ denote the 2×2 matrix with 1 at the $(\varepsilon, \varepsilon')$th place and 0 elsewhere. Thus the Pauli spin operators read

$$\sigma^x = E_{+-} + E_{-+} \qquad \sigma^y = -\mathrm{i}E_{+-} + \mathrm{i}E_{-+} \qquad \sigma^z = E_{++} - E_{--}\,.$$

In the tensor product $\cdots \otimes V_j \otimes V_{j+1} \otimes \cdots$ of $V_j \simeq \mathbb{C}^2$, we let σ_j^α, $E^{(j)}_{\varepsilon\varepsilon'}$ denote, respectively, the operators acting as σ^α or $E_{\varepsilon\varepsilon'}$ on the jth component and as identity elsewhere.

By a local operator we mean an element of the algebra generated by σ_j^α's. Any local operator is a linear combination of operators of the form $O = E^{(r)}_{\varepsilon'_r\varepsilon_r} E^{(r+1)}_{\varepsilon'_{r+1}\varepsilon_{r+1}} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}$ $(r \leqslant s)$. The correlation function of O is its expected value with respect to the ground-state eigenvector of the XXZ Hamiltonian. We conjecture that it is given by the following special value of G_n $(n = s - r + 1)$:

$$\langle E^{(r)}_{\varepsilon'_r\varepsilon_r} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle \stackrel{\mathrm{def}}{=} G_n(\overbrace{\beta+\pi\mathrm{i}, \ldots, \beta+\pi\mathrm{i}}^{n}, \overbrace{\beta, \ldots, \beta}^{n})_{-\varepsilon'_r,\ldots,-\varepsilon'_s,\varepsilon_s,\ldots,\varepsilon_r}\,. \tag{3.1}$$

We shall consider a slightly more general object

$$G_n(\beta_r+\pi\mathrm{i}, \ldots, \beta_s+\pi\mathrm{i}, \beta_s, \ldots, \beta_r)_{-\varepsilon'_r,\ldots,-\varepsilon'_s,\varepsilon_s,\ldots,\varepsilon_r}\,. \tag{3.2}$$

This corresponds to introducing spectral parameters β_j as inhomogeneity of the model (in the terminology of [13], the corresponding model is 'Z-invariant'). We shall denote (3.2) by

$$\langle E^{(r)}_{\varepsilon'_r\varepsilon_r} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle(\beta_r, \ldots, \beta_s)\,. \tag{3.3}$$

To see the specialization (3.2) is well defined, we note the following. In the general formula (2.18) the contour C for integration is such that $\beta_j + \mathrm{i}(2\pi n_1 + \pi n_2/\nu + \pi/2)$ $(n_1, n_2 \geqslant 0)$ (respectively $\beta_j - \mathrm{i}(2\pi n_1 + \pi n_2/\nu + \pi/2)$ $(n_1, n_2 \geqslant 0)$) is above (respectively below) C. When $\beta_j = \beta_k + \pi\mathrm{i}$, the contour C is pinched by $\beta_j - \pi\mathrm{i}/2$ and $\beta_k + \pi\mathrm{i}/2$. However, $Q_n(\alpha|\beta)$ has a zero at $\alpha_a = \beta_j - \pi\mathrm{i}/2$ for $\bar{a} > j$, and also at $\alpha_a = \beta_k + \pi\mathrm{i}/2$ for $\bar{a} < k$. Therefore, if $j < k$, there is no pinching by poles of the total integrand.

In order for (3.1) to make sense as a correlator, we must check the following property:

Proposition 3.1.

$$\langle E^{(r)}_{\varepsilon'_r\varepsilon_r} \cdots E^{(s-1)}_{\varepsilon'_{s-1}\varepsilon_{s-1}}(E^{(s)}_{++} + E^{(s)}_{--})\rangle(\beta_r, \ldots, \beta_{s-1}, \beta_s) = \langle E^{(r)}_{\varepsilon'_r\varepsilon_r} \cdots E^{(s-1)}_{\varepsilon'_{s-1}\varepsilon_{s-1}}\rangle(\beta_r, \ldots, \beta_{s-1}) \tag{3.4}$$

$$\langle (E^{(r)}_{++} + E^{(r)}_{--})E^{(r+1)}_{\varepsilon'_{r+1}\varepsilon_{r+1}} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle(\beta_r, \beta_{r+1}, \ldots, \beta_s) = \langle E^{(r+1)}_{\varepsilon'_{r+1}\varepsilon_{r+1}} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle(\beta_{r+1}, \ldots, \beta_s)\,. \tag{3.5}$$

Proof. Let us take $\beta_j = \beta_{j+1} + \pi\mathrm{i}$ in (2.4). Since

$$R(\pi\mathrm{i}) = \begin{pmatrix} 0 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 0 \end{pmatrix}$$

we find from (2.7) that

$$G_{n-1}(\beta_1, \ldots, \beta_{j-1}, \beta_{j+2}, \ldots, \beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{j-1},\varepsilon_{j+2},\ldots,\varepsilon_{2n}}$$
$$= \sum_{\varepsilon} G_n(\beta_1, \ldots, \beta_j, \beta_{j+1}, \ldots, \beta_{2n})_{\varepsilon_1,\ldots,\varepsilon,-\varepsilon,\ldots,\varepsilon_{2n}}\Big|_{\beta_j=\beta_{j+1}+\pi\mathrm{i}}\,.$$

Equation (3.4) is a direct consequence of this. The above equation together with (2.5) implies (since $\lambda = 2\pi$) that

$$G_{n-1}(\beta_2, \ldots, \beta_{2n-1})_{\varepsilon_2,\ldots,\varepsilon_{2n-1}} = \sum_{\varepsilon} G_n(\beta_1, \ldots, \beta_{2n})_{-\varepsilon,\varepsilon_2,\ldots,\varepsilon_{2n-1},\varepsilon}\Big|_{\beta_{2n}=\beta_1+\pi\mathrm{i}}.$$

From this follows (3.5). □

We now write down the integral formula for (3.2). Set $n = s-r+1$, and define $\bar{1}, \ldots, \bar{n}$ ($r \leqslant \bar{1} < \cdots < \bar{n} \leqslant s+n = r+2n-1$) by the following rule:

$$\{\bar{1}, \ldots, \bar{n}\} = \{j \mid r \leqslant j \leqslant s,\ \varepsilon'_j = -\} \cup \{j^* \mid r \leqslant j \leqslant s,\ \varepsilon_j = +\}$$

where $j^* = 2s+1-j$. We have then the following expression for (3.2).

$$\begin{aligned}
&\langle E^{(r)}_{\varepsilon'_r\varepsilon_r} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle(\beta_r, \ldots, \beta_s) \\
&\quad = \prod_{r\leqslant j<k\leqslant s} \frac{\sinh(\beta_j-\beta_k)}{\sinh\nu(\beta_j-\beta_k)} \int\cdots\int \prod_{l=1}^{n} \frac{\mathrm{d}\alpha_l}{2\pi} \prod_{1\leqslant l<l'\leqslant n} \frac{\sinh(\alpha_l-\alpha_{l'})}{\sinh\nu(\alpha_l-\alpha_{l'}-\pi\mathrm{i})} \\
&\qquad \times \prod_{r\leqslant\bar{l}\leqslant s}\left[\prod_{j=r}^{s} \frac{\mathrm{i}}{\sinh(\alpha_l-\beta_j+i0)} \prod_{r\leqslant j<\bar{l}} \sinh\nu(\alpha_l-\beta_j)\right. \\
&\qquad \times \left.\prod_{\bar{l}<j\leqslant s} \sinh\nu(-\alpha_l+\beta_j+\pi\mathrm{i})\right] \prod_{s+1\leqslant\bar{l}\leqslant s+n}\left[\prod_{j=r}^{s} \frac{-\mathrm{i}}{\sinh(\alpha_l-\beta_j-\mathrm{i}0)}\right. \\
&\qquad \times \left.\prod_{r\leqslant j<\bar{l}^*} \sinh\nu(-\alpha_l+\beta_j) \prod_{\bar{l}^*<j\leqslant s} \sinh\nu(\alpha_l-\beta_j+\pi\mathrm{i})\right].
\end{aligned} \tag{3.6}$$

Here the symbol $\alpha_l - \beta_j + \mathrm{i}0$ (respectively $\alpha_l - \beta_j - \mathrm{i}0$) indicates that the contour for α_l runs above (respectively below) β_j.

Let us put the formula in a form closer to the known result for $\nu = 0$, taking $r = 1$, $s = n$. We choose the integration contour C_+ for α_a ($1 \leqslant \bar{a} \leqslant n$) and C_- for α_a ($n+1 \leqslant \bar{a} \leqslant 2n$) in such a way that $\beta_j + \pi\mathrm{i}$ (respectively β_j) ($1 \leqslant j \leqslant n$) are above (respectively below) C_+ and β_j (respectively $\beta_j - \pi\mathrm{i}$) ($1 \leqslant j \leqslant n$) are above (respectively below) C_-. (Here the contours are directed from $-\infty$ to ∞, as opposed to the $C^{\pm}$ in pp 122–3 of [2].)

Set $A' = \{j \mid \varepsilon'_j = -\}$ and $A = \{j \mid \varepsilon_j = +\}$. We suppose that $\sharp(A') + \sharp(A) = n$ since otherwise $\langle E^{(1)}_{\varepsilon'_1\varepsilon_1} \cdots E^{(n)}_{\varepsilon'_n\varepsilon_n}\rangle(\beta_1, \ldots, \beta_n) = 0$. We define a mapping

$$a \in \{1, \ldots, n\} = A_+ \sqcup A_- \to \bar{a} \in \{1, \ldots, n\} \tag{3.7}$$

by the condition that (i) $\{\bar{a} \mid a \in A_+\} = A'$, $\{\bar{a} \mid a \in A_-\} = A$; (ii) if $a, b \in A_+$ and $a < b$ then $\bar{a} < \bar{b}$; (iii) if $a, b \in A_-$ and $a < b$ then $\bar{a} > \bar{b}$. In other words, the +'s in the sequence $-\varepsilon'_1, \ldots, -\varepsilon'_n, \varepsilon_n, \ldots, \varepsilon_1$ are $-\varepsilon'_{\bar{1}}, \ldots, -\varepsilon'_{\bar{s}'}, \varepsilon_{\bar{s}}, \ldots, \varepsilon_{\bar{1}}$ where $s = \sharp(A_-) = n - s'$. Then we have

$$\begin{aligned}
&G_n(\beta_1+\pi\mathrm{i}/2, \ldots, \beta_n+\pi\mathrm{i}/2, \beta_n-\pi\mathrm{i}/2, \ldots, \beta_1-\pi\mathrm{i}/2)_{-\varepsilon'_1,\ldots,-\varepsilon'_n,\varepsilon_n,\ldots,\varepsilon_1} \\
&\quad = (-1)^{\sum_{a\in A_+}\bar{a}+\sum_{a\in A_-}\bar{a}+\sharp(A_-)+n(n-1)/2} \prod_{1\leqslant j<k\leqslant n} \frac{\sinh(\beta_j-\beta_k)}{\sinh\nu(\beta_j-\beta_k)} \\
&\qquad \times \prod_{a\in A_+} \int_{C_+} \frac{\mathrm{d}\alpha_a}{2\pi\mathrm{i}\sinh\nu(\alpha_a-\beta_{\bar{a}})} \prod_{a\in A_-} \int_{C_-} \frac{\mathrm{d}\alpha_a}{2\pi\mathrm{i}\sinh\nu(\alpha_a-\beta_{\bar{a}})} \\
&\qquad \times \prod_{1\leqslant a<b\leqslant n} \frac{\sinh(\alpha_a-\alpha_b)}{\sinh\nu(\alpha_a-\alpha_b-\pi\mathrm{i})} \prod_{1\leqslant a,j\leqslant n} \frac{\sinh\nu(\alpha_a-\beta_j)}{\sinh(\alpha_a-\beta_j)}
\end{aligned}$$

$$\times \prod_{\substack{a\in A_+\\ j>\bar{a}}} \frac{\sinh \nu(\beta_j - \alpha_a + \pi \mathrm{i})}{\sinh \nu(\beta_j - \alpha_a)} \prod_{\substack{a\in A_-\\ j>\bar{a}}} \frac{\sinh \nu(\alpha_a - \beta_j + \pi \mathrm{i})}{\sinh \nu(\alpha_a - \beta_j)} . \tag{3.8}$$

In the limit $\nu = 0$ we recover the integral formula for the XXX correlation functions ([2, 11, 12]).

4. Nearest-neighbour correlator

In this section we examine the simplest cases of the general formula for the correlators proposed in the previous section.

First consider the case $G_1(\beta_1, \beta_2)$. Taking $n = 1$ in formula (2.28), we immediately find the following.

Proposition 4.1.

$$G_1(\beta_1, \beta_2)_{-+} = G_1(\beta_1, \beta_2)_{+-} = \frac{1}{2\nu} \frac{\rho(\beta_1 - \beta_2)}{\rho(\pi \mathrm{i})} \frac{\sinh \frac{1}{2}\nu(\beta_1 - \beta_2 - \pi \mathrm{i})}{\sinh \frac{1}{2}(\beta_1 - \beta_2 - \pi \mathrm{i})} .$$

In particular, by setting $\beta_1 = \beta_2 + \pi \mathrm{i}$, we have

$$\langle E^{(1)}_{++} \rangle = \langle E^{(1)}_{--} \rangle = \tfrac{1}{2} . \tag{4.1}$$

Next let us take $n = 2$ in (2.28).

Proposition 4.2. Assuming $0 < \nu < \frac{1}{2}$, we have

$$\begin{aligned} G_2(\beta_1, \ldots, \beta_4)_{++--} = {}& \frac{\prod_{j<k} \rho(\beta_j - \beta_k)}{\rho(0)^2 \rho(\pi \mathrm{i})^4} \frac{\mathrm{e}^{-\sum_{j=1}^4 \beta_j/2}}{2\pi \nu^2 \sum_{j=1}^4 \mathrm{e}^{-\beta_j}} \\ & \times \int \mathrm{d}\alpha\, \mathrm{e}^{\alpha} \prod_{j=1}^4 \varphi(\alpha - \beta_j)\, \sinh \nu\big(-\alpha + \beta_3 + \pi \mathrm{i}/2\big) \sinh \nu\big(-\alpha + \beta_4 + \pi \mathrm{i}/2\big) \\ & \times \Big[\sinh \nu\big(-\alpha + \beta_2 + \pi \mathrm{i}/2\big) \sinh \nu\big(\alpha + (\beta_2 - \beta_1 - \beta_3 - \beta_4)/2 - 3\pi \mathrm{i}/2\big) \\ & - \sinh \nu\big(\alpha - \beta_1 + \pi \mathrm{i}/2\big) \sinh \nu\big(\alpha + (\beta_1 - \beta_2 - \beta_3 - \beta_4)/2 - \pi \mathrm{i}/2\big) \Big] . \end{aligned} \tag{4.2}$$

The integral is taken along a path from $-\infty$ to $+\infty$ such that $-\pi/2 < \mathrm{Im}(\alpha - \beta_j) < \pi/2$ for all j.

Here we have used the relation $\rho(0)\rho(\pi \mathrm{i}) = -1/(4\sqrt{\nu})$ which follows from (2.12), (2.13) and $S_2(\pi|2\pi, \pi/\nu) = \sqrt{2}$.

Upon specialization $(\beta_1, \ldots, \beta_4) = (\beta + \pi \mathrm{i}, \beta + \pi \mathrm{i}, \beta, \beta)$, this integral can be processed further. After a chain of steps detailed in appendix D, we obtain the following result.

Proposition 4.3. We have

$$\begin{aligned} \langle E^{(1)}_{--} E^{(2)}_{--} \rangle &= G_2(\beta + \pi \mathrm{i}, \beta + \pi \mathrm{i}, \beta, \beta)_{++--} \\ &= \frac{1}{\pi^2 \sin \pi\nu} \frac{\mathrm{d}}{\mathrm{d}\nu} \left(\sin \pi\nu \int_0^\infty \frac{\sinh(1-\nu)t}{\sinh t \cosh \nu t} \mathrm{d}t \right) + \frac{1}{2} . \end{aligned} \tag{4.3}$$

We note that, since both sides are holomorphic with respect to ν for $0 < \mathrm{Re}\, \nu < 1$, equation (4.3) is valid without the restriction $0 < \nu < \frac{1}{2}$.

We now compare formulae (4.1) and (4.3) with known answers. For this purpose let us quote from [3] the results concerning the XXZ model which are relevant to the following discussion.

The XXZ model for a periodic chain of circumference N is given by the Hamiltonian

$$H = -\tfrac{1}{2}\sum_{j=1}^{N}\bigl(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z\bigr)\,. \tag{4.4}$$

This Hamiltonian (4.4) is associated with the six-vertex model with the Boltzmann weights ([3], equations (8.8) and (8.9))

$$a = \sin\frac{\mu - w}{2} \qquad b = \sin\frac{\mu + w}{2} \qquad c = \sin\mu\,. \tag{4.5}$$

Denoting by $T(w)$ the transfer matrix of the periodic system with N columns, we have

$$\frac{\mathrm{d}}{\mathrm{d}w}\log T(w)\big|_{w=-\mu} = -\frac{1}{2\sin\mu}\left(H + \frac{N}{2}\cos\mu\right) \tag{4.6}$$

where Δ is related to μ via

$$\Delta = -\cos\mu\,. \tag{4.7}$$

The gapless regime $|\Delta| \leqslant 1$ corresponds to μ being real.

For a local operator O, let $\langle\mathrm{vac}|O|\mathrm{vac}\rangle$ denote its ground-state average (in the limit $N \to \infty$). Since in the gapless regime the vacuum $|\mathrm{vac}\rangle$ is invariant under the $+ \leftrightarrow -$ symmetry, one must have

$$\langle\mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle = \langle\mathrm{vac}|\bigl(E_{++}^{(1)} - E_{--}^{(1)}\bigr)|\mathrm{vac}\rangle = 0\,.$$

Along with $1 = \langle\mathrm{vac}|\bigl(E_{++}^{(1)} + E_{--}^{(1)}\bigr)|\mathrm{vac}\rangle$, this means

$$\langle\mathrm{vac}|E_{++}^{(1)}|\mathrm{vac}\rangle = \langle\mathrm{vac}|E_{--}^{(1)}|\mathrm{vac}\rangle = \tfrac{1}{2}\,. \tag{4.8}$$

Our formula (4.1) is consistent with this.

In the limit $N \to \infty$, the free energy per site f is given by ([3], equation (8.8.17))

$$-\frac{f}{kT} = \log a + \int_{-\infty}^{\infty}\frac{\sinh(\mu + w)x\,\sinh(\pi - \mu)x}{2x\sinh\pi x\cosh\mu x}\,\mathrm{d}x\,. \tag{4.9}$$

It follows from the relation (4.6) that the ground-state energy per site e_0 of the XXZ chain is

$$e_0 = -\tfrac{1}{2}\cos\mu - 2\sin\mu\frac{\partial}{\partial w}\left(-\frac{f}{kT}\right)\bigg|_{w=-\mu}\,. \tag{4.10}$$

Differentiating e_0 with respect to Δ, we can obtain the nearest-neighbour correlator for the σ^z operators:

$$\langle\mathrm{vac}|\sigma_1^z\sigma_2^z|\mathrm{vac}\rangle = -2\frac{\mathrm{d}e_0}{\mathrm{d}\Delta} = -\frac{2}{\sin\mu}\frac{\mathrm{d}e_0}{\mathrm{d}\mu}\,.$$

Inserting (4.9) into (4.10), we find the following expression for this quantity:

$$\langle\mathrm{vac}|\sigma_1^z\sigma_2^z|\mathrm{vac}\rangle = 1 + \frac{4}{\sin\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\sin\mu\int_0^{\infty}\frac{\sinh(\pi - \mu)x}{\sinh\pi x\cosh\mu x}\,\mathrm{d}x\right). \tag{4.11}$$

On the other hand, in view of (4.8) we have

$$\langle\mathrm{vac}|\sigma_1^z\sigma_2^z|\mathrm{vac}\rangle = \langle\mathrm{vac}|(1 - 2E_{--}^{(1)})(1 - 2E_{--}^{(2)})|\mathrm{vac}\rangle = 4\langle\mathrm{vac}|E_{--}^{(1)}E_{--}^{(2)}|\mathrm{vac}\rangle - 1\,.$$

Therefore, formula (4.3) agrees with (4.11) with the identification $\mu = \pi\nu$.

5. The XY limit

In this section we study the integral formula for the correlation functions (3.6) at a special value of the parameter $\nu = \frac{1}{2}$. This is the case where the XXZ chain reduces to the XY chain $\Delta = 0$. It is well known that the XY chain is equivalent to the two-dimensional Ising model. To be more precise, the XXZ model with $\Delta = 0$ corresponds to the critical Ising model. In this case, diagonalizing the transfer matrix in terms of free fermions, one can calculate the correlation functions directly in the presence of arbitrary spectral parameters. The diagonalization is worked out in appendix F. Here we show that the formulae thus obtained give the same result as the integral formula (3.6).

We shall consider the function

$$\langle E^{(r)}_{\varepsilon'_r\varepsilon_r} E^{(r+1)}_{\varepsilon'_{r+1}\varepsilon_{r+1}} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle(\beta_r, \beta_{r+1}, \ldots, \beta_s)$$

given by (3.3). In order to simplify the presentation, we shall take all β_j's to be real throughout this section. (This is a matter of convenience and not actually a restriction. The final formulae are valid as meromorphic functions in β_j's.)

A special feature about $\nu = \frac{1}{2}$ is that (3.6) becomes a determinant.

Proposition 5.1. If $\nu = \frac{1}{2}$, we have

$$\langle E^{(r)}_{\varepsilon'_r\varepsilon_r} \cdots E^{(s)}_{\varepsilon'_s\varepsilon_s}\rangle(\beta_r, \ldots, \beta_s) = \prod_{r\leqslant j<k\leqslant s} 2\mathrm{i}\cosh\tfrac{1}{2}(\beta_j - \beta_k) \times \det(I_{k\bar{k}'})_{1\leqslant k,k'\leqslant n}\,. \tag{5.1}$$

Here $n = s - r + 1$, the I_{kl} are given for $r \leqslant l \leqslant s$ by

$$I_{kl} = 2\mathrm{i}^{2r-s+1-l}\int \frac{\mathrm{d}\alpha}{2\pi}\, \mathrm{e}^{(k-(n+1)/2)\alpha} \prod_{j=r}^{l} \frac{1}{2\cosh\frac{1}{2}(\alpha - \beta_j)} \prod_{j=l}^{s} \frac{1}{2\sinh\frac{1}{2}(\alpha - \beta_j + \mathrm{i}0)} \tag{5.2}$$

and for $s + 1 \leqslant l \leqslant s + n$

$$I_{kl} = \bar{I}_{kl^*}$$

with the bar denoting the complex conjugate. In (5.2), the integration contour is a line above the real axis, as indicated by the symbol $+\,\mathrm{i}0$.

Proof. Specializing the formula (3.6) to $\nu = \frac{1}{2}$ we find

$$\begin{aligned}
&\prod_{r\leqslant j<k\leqslant s} 2\cosh\frac{\beta_j - \beta_k}{2} \int\cdots\int \prod_{l=1}^{n}\frac{\mathrm{d}\alpha_l}{2\pi} \prod_{l<l'} 2\mathrm{i}\sinh\tfrac{1}{2}(\alpha_l - \alpha_{l'}) \\
&\qquad \times \prod_{r\leqslant\bar{l}\leqslant s}\left[2\mathrm{i}^{n+s-\bar{l}} \prod_{j=r}^{\bar{l}} \frac{1}{2\cosh\frac{1}{2}(\alpha_l - \beta_j)} \prod_{j=\bar{l}}^{s} \frac{1}{2\sinh\frac{1}{2}(\alpha_l - \beta_j + \mathrm{i}0)}\right] \\
&\qquad \times \prod_{s+1\leqslant\bar{l}\leqslant s+n}\left[2\mathrm{i}^{-r-1+\bar{l}^*} \prod_{j=r}^{\bar{l}^*} \frac{1}{2\cosh\frac{1}{2}(\alpha_l - \beta_j)} \prod_{j=\bar{l}^*}^{s} \frac{1}{2\sinh\frac{1}{2}(\alpha_l - \beta_j - \mathrm{i}0)}\right].
\end{aligned}$$

Inserting

$$\prod_{l<l'} 2\mathrm{i}\sinh\tfrac{1}{2}(\alpha_l - \alpha_{l'}) = \mathrm{i}^{-n(n-1)/2}\, \mathrm{e}^{-\frac{n+1}{2}(\alpha_1+\cdots+\alpha_n)} \det(\mathrm{e}^{k\alpha_{k'}})_{1\leqslant k,k'\leqslant n}$$

we obtain the right-hand side of (5.1). □

In appendix E we evaluate the integral (5.2) explicitly (see equation (E.2)).

We now proceed to the calculation of correlation functions of the fermion operators

$$\psi_m^* = \cdots \sigma_{m-2}^z \sigma_{m-1}^z \sigma_m^+ \qquad \psi_m = \cdots \sigma_{m-2}^z \sigma_{m-1}^z \sigma_m^- \,.$$

We shall consider only monomials consisting of an *even* number of such operators. They are local operators in the sense of section 3. For instance,

$$\psi_m \psi_l^* = \begin{cases} \sigma_m^- \sigma_{m+1}^z \cdots \sigma_{l-1}^z \sigma_l^+ & (m < l) \\ E_{--}^{(m)} & (m = l) \\ \sigma_l^+ \sigma_{l+1}^z \cdots \sigma_{m-1}^z \sigma_m^- & (m > l) \,. \end{cases}$$

Clearly the function $\langle O \rangle$ for a monomial O is 0 (see equation (3.3)) unless it consists of the same number of ψ's and ψ^*'s.

The following two propositions will be proved in appendix E.

Proposition 5.2.

$$\langle \psi_{m_1} \cdots \psi_{m_k} \psi_{l_k}^* \cdots \psi_{l_1}^* \rangle = \det \left(\langle \psi_{m_j} \psi_{l_i}^* \rangle \right)_{1 \leqslant j, i \leqslant k} \,. \tag{5.3}$$

Proposition 5.3.

$$\begin{aligned} \langle \psi_m \psi_l^* \rangle &= (-1)^{m+l} \langle \psi_m^* \psi_l \rangle \\ &= -\frac{\mathrm{i}^{l-m+1}}{\pi} (B_m B_l)^{1/2} \sum_{j=m}^{l} \beta_j \frac{\prod_{i=m+1}^{l-1} (B_j + B_i)}{\prod_{\substack{i=m \\ i \neq j}}^{l} (B_j - B_i)} \qquad (m < l) && (5.4) \\ &= \tfrac{1}{2} \qquad (m = l) \,. && (5.5) \end{aligned}$$

Formulae (5.4) and (5.5) give the same results for the corresponding quantities (F.16) $\langle \mathrm{vac} | \psi_m \psi_l^* | \mathrm{vac} \rangle$ obtained directly by diagonalizing the Hamiltonian (see appendix F). In general, the multiple correlators of the fermions are given by applying Wick's theorem. Since $\langle \mathrm{vac} | \psi_m \psi_l | \mathrm{vac} \rangle = \langle \mathrm{vac} | \psi_m^* \psi_l^* | \mathrm{vac} \rangle = 0$, the result is given as a determinant in the same way as (5.3). Any local operator (i.e. a finite linear combination of monomials in σ_j^α's) can also be written as a linear combination of monomials of the fermions. Therefore, we can state

Proposition 5.4. For an arbitrary local operator O, $\langle O \rangle = \langle \mathrm{vac} | O | \mathrm{vac} \rangle$ holds.

6. Discussions

Before concluding the paper, let us touch upon previous works on the qKZ equation with $|q| = 1$. In [14] Smirnov introduced and solved a system of difference equations for the form factors of local operators in the sine–Gordon theory. His equations are the same as (2.4) and (2.5) in this paper except that $S = -R$ is used (see p 29 in [14]) and that $\lambda = -2\pi$ instead of $\lambda > 0$ (the case relevant to the correlation function is $\lambda = 2\pi$). There is a significant difference between (2.6) in this paper and equation (16) (p 11) in Smirnov's. The latter requires that the solution has a simple pole at the point $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$, while the former requires that the solution is regular there. The physical origin of this difference is that in Smirnov's case the poles are the annihilation poles of the form factors while in our case (2.6) is the normalization of the correlation functions (see proposition 3.1).

There are other mathematical differences between Smirnov's formula and ours. The number of integrations is n-fold in our formula in contrast to the $(n-1)$-fold integrals in Smirnov's. Since the integration can be carried out once (see (2.28)), this difference is rather superficial. The significant difference is that in Smirnov's formula the $(n-1)$-fold integral reduces to the determinant of an $(n-1)\times(n-1)$ matrix with entries given by integrals with respect to a single variable. This is not the case in our formula (except for $\nu=\frac{1}{2}$—the case of the XY model). This lack of determinantal structure is already noted by Nakayashiki [12], where the special case $\nu=0$ was studied.

In a recent paper [15], Smirnov has constructed an affluent family of solutions that corresponds to a family of local operators in the sine–Gordon theory. In our case, the structure of the total space of solutions is absolutely unknown. In this connection, let us mention an open problem: to show that our integrals satisfy $G_n(\beta_1,\ldots,\beta_{2n})_{\varepsilon_1,\ldots,\varepsilon_{2n}} = G_n(\beta_1,\ldots,\beta_{2n})_{-\varepsilon_1,\ldots,-\varepsilon_{2n}}$. A direct verification seems difficult, and we suspect that it should follow from the uniqueness of the solution satisfying certain analyticity and asymptotics.

In fact, the form factors and the correlation functions are closely related. As was discussed in [2, 16], in the regime $\Delta<-1$, the former are represented by the type-II vertex operators and the latter by the type-I vertex operators. The type-I vertex operators generate a family of solutions to the form factor equations, and vice versa. In the sine–Gordon theory, this viewpoint was explored by Lukyanov [10]. Lukyanov has introduced the appropriate commutation relations for the vertex operators, and has given a bosonization of the sine–Gordon theory with a cut-off parameter. Though we have not checked the details, it seems likely that the integral formula for G_n in this paper is derivable from Lukyanov's bosonization after taking the cut-off parameter to infinity.

In the approach of this paper, the role of the quantum affine algebra $U_q(\widehat{sl}_2)$ is unclear. In the $\Delta<-1$ regime, the free energy, the excitation spectrum and the correlation functions depend on the spectral parameters β_j through $\zeta_j=\mathrm{e}^{-\nu\beta_j}$. In the gapless regime, these quantities are single-valued only in β_j and the period $2\pi\mathrm{i}/\nu$ is lost. What is the implication of this fact in the representation theory? This is an interesting question to be asked.

Acknowledgments

We thank Sergei Lukyanov, Atsushi Nakayashiki and Feodor Smirnov for valuable discussions. We wish to express our sorrow at the death of Claude Itzykson. We have always liked his nice lectures given in his fascinating voice ever since we first met him in San Francisco in 1979.

Appendix A. Multiple gamma functions

The multiple gamma and sine functions were introduced by Barnes [17, 18], Shintani [19] and Kurokawa [20]. Here we follow the notation of [20]. In what follows we fix an r-tuple of complex numbers $\underline{\omega}=(\omega_1,\ldots,\omega_r)$. For simplicity we shall assume that $\mathrm{Re}\,\omega_i>0$. We set $\underline{n}\cdot\underline{\omega}=n_1\omega_1+\cdots+n_r\omega_r$ $(\underline{n}=(n_1,\ldots,n_r))$, $|\underline{\omega}|=\omega_1+\cdots+\omega_r$.

The multiple gamma and associated functions are defined as follows.

Multiple Hurwitz zeta function

$$\zeta_r(s,x|\underline{\omega}) = \sum_{n_1,\ldots,n_r\geqslant 0}(\underline{n}\cdot\underline{\omega}+x)^{-s}\,. \tag{A.1}$$

Multiple gamma function

$$\Gamma_r(x|\underline{\omega}) = \exp\big(\zeta_r'(0,x|\underline{\omega})\big) \qquad \left({}'=\frac{\partial}{\partial s}\right). \tag{A.2}$$

Multiple digamma function

$$\psi_r(x|\underline{\omega}) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma_r(x|\underline{\omega})\,. \tag{A.3}$$

Multiple sine function

$$S_r(x|\underline{\omega}) = \Gamma_r(x|\underline{\omega})^{-1}\Gamma_r(|\underline{\omega}| - x|\underline{\omega})^{(-1)^r}. \tag{A.4}$$

Multiple Bernoulli polynomials

$$\frac{t^r e^{xt}}{\prod_{i=1}^r (e^{\omega_i t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\underline{\omega}). \tag{A.5}$$

When $r = 1$, they are related to the ordinary gamma and other functions via

$$\zeta_1(s, x|\omega_1) = \omega_1^{-s}\zeta\left(s, \frac{x}{\omega_1}\right)$$
$$\Gamma_1(x|\omega_1) = \omega_1^{x/\omega_1 - \frac{1}{2}} \frac{\Gamma(x/\omega_1)}{\sqrt{2\pi}}$$
$$\psi_1(x|\omega_1) = \frac{1}{\omega_1}\left(\psi\left(\frac{x}{\omega_1}\right) + \log\omega_1\right)$$
$$S_1(x|\omega_1) = 2\sin\left(\frac{\pi x}{\omega_1}\right).$$

Here we list the basic properties of these functions.

Difference equations. Set $\underline{\omega}(i) = (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_r)$.

$$\zeta_r(s, x + \omega_i|\underline{\omega}) - \zeta_r(s, x|\underline{\omega}) = \zeta_{r-1}(s, x|\underline{\omega}(i)) \tag{A.6}$$
$$\frac{\Gamma_r(x + \omega_i|\underline{\omega})}{\Gamma_r(x|\underline{\omega})} = \frac{1}{\Gamma_{r-1}(x|\underline{\omega}(i))} \tag{A.7}$$
$$\frac{S_r(x + \omega_i|\underline{\omega})}{S_r(x|\underline{\omega})} = \frac{1}{S_{r-1}(x|\underline{\omega}(i))} \tag{A.8}$$
$$B_{r,n}(x + \omega_i|\underline{\omega}) - B_{r,n}(x|\underline{\omega}) = nB_{r-1,n-1}(x|\underline{\omega}(i)). \tag{A.9}$$

Analyticity. As a function of s, $\zeta_r(s, x|\underline{\omega})$ is continued meromorphically on the whole complex plane and is holomorphic except for simple poles at $s = 1, \ldots, r$. We have

$$\zeta_r(n, x|\underline{\omega}) = \frac{(-1)^n}{(n-1)!}\psi_r^{(n-1)}(x|\underline{\omega}) \qquad (n > r)$$
$$\zeta_r(-n, x|\underline{\omega}) = (-1)^r \frac{n!}{(n+r)!} B_{r,n+r}(x|\underline{\omega}) \qquad (n \geqslant 0)$$
$$\lim_{s\to n}(s-n)\zeta_r(s, x|\underline{\omega}) = (-1)^{n-r}\frac{B_{r,r-n}(x|\underline{\omega})}{(n-1)!(r-n)!} \qquad (r \geqslant n \geqslant 1).$$

$\Gamma_r(x|\underline{\omega})^{-1}$ is an entire function of x. $\Gamma_r(x|\underline{\omega})$ is meromorphic with poles at $x = \underline{n}\cdot\underline{\omega}$ $(n_1, \ldots, n_r \leqslant 0)$.

$S_r(x|\underline{\omega})$ is entire in x when r is odd, and is meromorphic when r is even. Its zeros and poles are given by

r odd	zeros at $x = \underline{n}\cdot\underline{\omega}$	$(n_1, \ldots, n_r \geqslant 1$ or $n_1, \ldots, n_r \leqslant 0)$
r even	zeros at $x = \underline{n}\cdot\underline{\omega}$	$(n_1, \ldots, n_r \leqslant 0)$
	poles at $x = \underline{n}\cdot\underline{\omega}$	$(n_1, \ldots, n_r \geqslant 1)$.

All zeros and poles are simple if $\underline{n}\cdot\underline{\omega}$'s do not overlap.

Integral representations. If $\mathrm{Re}\, x > 0$ then

$$\zeta_r(s, x|\underline{\omega}) = -\Gamma(1-s)\int_C \frac{\mathrm{e}^{-xt}(-t)^{s-1}}{\prod_{i=1}^{r}(1-\mathrm{e}^{-\omega_i t})}\frac{\mathrm{d}t}{2\pi\mathrm{i}}$$

$$\log\Gamma_r(x|\underline{\omega}) = \gamma\frac{(-1)^r}{r!}B_{r,r}(x|\underline{\omega}) + \int_C \frac{\mathrm{e}^{-xt}\log(-t)}{\prod_{i=1}^{r}(1-\mathrm{e}^{-\omega_i t})}\frac{\mathrm{d}t}{2\pi\mathrm{i}t}$$

$$\psi_r(x|\underline{\omega}) = \gamma\frac{(-1)^r}{r!}B'_{r,r}(x|\underline{\omega}) - \int_C \frac{\mathrm{e}^{-xt}\log(-t)}{\prod_{i=1}^{r}(1-\mathrm{e}^{-\omega_i t})}\frac{\mathrm{d}t}{2\pi\mathrm{i}}$$

where $\gamma =$ Euler's constant and $\Gamma(x)$ denotes the ordinary gamma function. The contour C is shown in figure A1.

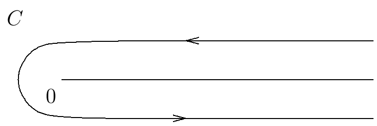


Figure A1. The contour C.

Asymptotic expansion. Assume $\omega_1, \ldots, \omega_r > 0$. Then for any $N \geqslant 1$ we have

$$\log\Gamma_r(z|\underline{\omega}) = (-1)^{r-1}\sum_{k=0}^{r}\frac{B_{r,r-k}(0)}{(r-k)!}\frac{z^k}{k!}\left(\log z + \gamma - \sum_{j=1}^{k}\frac{1}{j}\right) + \gamma\zeta_r(0,z)$$
$$+\sum_{n=1}^{N}(-1)^{n-r}\frac{(n-1)!}{(n+r)!}B_{r,r+n}(0)z^{-n} + \mathrm{o}(z^{-N}) \qquad \text{(A.10)}$$

as $z \to \infty$ in the angular domain $|\,\mathrm{Arg}(z-x)| \leqslant \pi - \epsilon$, where $x > 0$ and $0 < \epsilon < \pi$.

The case $r = 2$ is of special interest to us. In this case the following formulae hold.

$$\log S_2(x|\underline{\omega}) = \int_C \frac{\sinh\,(x-(\omega_1+\omega_2)/2)\,t}{2\sinh\omega_1 t/2\sinh\omega_2 t/2}\log(-t)\frac{\mathrm{d}t}{2\pi\mathrm{i}t} \qquad (0 < \mathrm{Re}\, x < \omega_1+\omega_2)$$

$$\frac{S_2(x+\omega_1|\underline{\omega})}{S_2(x|\underline{\omega})} = \frac{1}{2\sin\pi x/\omega_2} \qquad \text{(A.11)}$$

$$S_2(x|\underline{\omega})S_2(-x|\underline{\omega}) = -4\sin\frac{\pi x}{\omega_1}\sin\pi x/\omega_2 \qquad \text{(A.12)}$$

$$S_2(x|\underline{\omega}) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}x + \mathrm{O}(x^2) \qquad (x \to 0) \qquad \text{(A.13)}$$

$$S_2(\omega_1|\underline{\omega}) = \sqrt{\frac{\omega_2}{\omega_1}} \qquad S_2\left(\frac{\omega_1}{2}\middle|\underline{\omega}\right) = \sqrt{2} \qquad S_2\left(\frac{\omega_1+\omega_2}{2}\middle|\underline{\omega}\right) = 1\,.$$

In addition, as $x \to \infty$ ($\pm\,\mathrm{Im}\, x > 0$), we have

$$\log S_2(x|\underline{\omega}) = \pm\pi\mathrm{i}\left(\frac{x^2}{2\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}x - \frac{1}{12}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+3\right)\right) + \mathrm{o}(1)$$
$$\log S_2(a+x|\underline{\omega})S_2(a-x|\underline{\omega}) = \pm\pi\mathrm{i}\frac{(2a-\omega_1-\omega_2)}{\omega_1\omega_2}x + \mathrm{o}(1)\,. \qquad \text{(A.14)}$$

Appendix B. Proof of difference equations for G_n

Here we prove that the integral formula (2.18) possesses the required properties (2.4)–(2.6).

Proof of (2.4). Let $\bar{G}_n = G_n/\rho_n$, $\rho_n = \prod_{j<k}\rho(\beta_j - \beta_k)$. Because of (2.8), (2.4) reduces to the same equation for $\bar{G}_n$ wherein R is replaced by $\bar{R}$.

There are four cases to consider: $(\varepsilon_j, \varepsilon_{j+1}) = (-,-)$, $(+,-)$, $(-,+)$ and $(+,+)$.

- Case $(\varepsilon_j, \varepsilon_{j+1}) = (-,-)$. We are to show that

$$\bar{G}_n(\dots, \beta_{j+1}, \beta_j, \dots)_{\dots--\dots} = \bar{G}_n(\dots, \beta_j, \beta_{j+1}, \dots)_{\dots--\dots} . \tag{B.1}$$

  This is obvious because the integrand of $\bar{G}_n$ is symmetric with respect to β_j and β_{j+1} if $(\varepsilon_j, \varepsilon_{j+1}) = (-,-)$.
- Case $(\varepsilon_j, \varepsilon_{j+1}) = (-,+)$. Suppose that $\bar{a} = j+1$, and set $\alpha = \alpha_a$. Comparing the integrands of $\bar{G}_n(\dots, \beta_{j+1}, \beta_j, \dots)_{\dots+-\dots}$ and $\bar{G}_n(\dots, \beta_j, \beta_{j+1}, \dots)_{\dots\pm\mp\dots}$, we see that the desired equality follows from

$$\bar{b}(\beta_j - \beta_{j+1}) \sinh \nu(\alpha - \beta_j + \pi \mathrm{i}/2) + \bar{c}(\beta_j - \beta_{j+1}) \sinh \nu(\beta_{j+1} - \alpha + \pi \mathrm{i}/2)$$
$$= \sinh \nu(\beta_j - \alpha + \pi \mathrm{i}/2) .$$

- Case $(\varepsilon_j, \varepsilon_{j+1}) = (+,-)$. This is similar to the case $(\varepsilon_j, \varepsilon_{j+1}) = (-,+)$.
- Case $(\varepsilon_j, \varepsilon_{j+1}) = (+,+)$. We are to show that

$$\bar{G}_n(\dots, \beta_{j+1}, \beta_j, \dots)_{\dots++\dots} = \bar{G}_n(\dots, \beta_j, \beta_{j+1}, \dots)_{\dots++\dots} . \tag{B.2}$$

  Suppose that $\bar{a} = j$, and set $\alpha = \alpha_a$ and $\alpha' = \alpha_{a+1}$. Apart from the factors that are symmetric with respect to β_j and β_{j+1} and antisymmetric with respect to α and α', the integrand of the LHS of (B.2) contains

$$\frac{\sinh \nu(\alpha' - \beta_{j+1} + \pi \mathrm{i}/2) \sinh \nu(\beta_j - \alpha + \pi \mathrm{i}/2)}{\sinh \nu(\alpha - \alpha' - \pi \mathrm{i})} . \tag{B.3}$$

  Antisymmetrizing it with respect to the variables α and α', we obtain an expression that is symmetric with respect to β_j and β_{j+1}. Therefore, we have (B.2).

Proof of (2.5). Because of (2.9), the equality (2.5) is equivalent to

$$\bar{G}_n(\beta_1, \dots, \beta_{2n-1}, \beta_{2n} - \mathrm{i}\lambda)_{\varepsilon_1\cdots\varepsilon_{2n}} = \bar{G}_n(\beta_{2n}, \beta_1, \dots, \beta_{2n-1})_{\varepsilon_{2n}\varepsilon_1\cdots\varepsilon_{2n-1}} . \tag{B.4}$$

If $\varepsilon_{2n} = -$, then $\bar{n} \neq 2n$. In this case, the integrands of the LHS and the RHS coincide because of (2.11):

$$\varphi(\alpha_a - \beta_{2n} + \mathrm{i}\lambda) \sinh \nu(\beta_{2n} - \alpha_a + \pi \mathrm{i}/2 - \mathrm{i}\lambda) = \varphi(\alpha_a - \beta_{2n}) \sinh \nu(\alpha_a - \beta_{2n} + \pi \mathrm{i}/2) . \tag{B.5}$$

If $\varepsilon_{2n} = +$, then $\bar{n} = 2n$. We make the following change of integration variables:

$$\alpha_n \to \alpha_n - \mathrm{i}\lambda \text{ in the LHS}$$
$$\begin{cases} \alpha_1 \to \alpha_n \\ \alpha_2 \to \alpha_1 \\ \dots \\ \alpha_n \to \alpha_{n-1} \qquad \text{in the RHS} . \end{cases} \tag{B.6}$$

Then the integrands become the same by virtue of (2.11),

$$\varphi(\alpha_n - \mathrm{i}\lambda - \beta_j) \sinh \nu(\alpha_n - \mathrm{i}\lambda - \beta_j + \pi \mathrm{i}/2) = \varphi(\alpha_n - \beta_j) \sinh \nu(\beta_j - \alpha_n + \pi \mathrm{i}/2) \tag{B.7}$$

and (2.14), (2.16),

$$\frac{\psi(\alpha_a - \alpha_n + \mathrm{i}\lambda)}{\sinh \nu(\alpha_a - \alpha_n + \mathrm{i}\lambda - \pi \mathrm{i})} = \frac{\psi(\alpha_n - \alpha_a)}{\sinh \nu(\alpha_n - \alpha_a - \pi \mathrm{i})} . \tag{B.8}$$

We must also check that the contours for the LHS and the RHS are the same. Consider the contour $\tilde{C}_a$ corresponding to α_a except for the case when $\varepsilon_{2n} = +$ and $a = n$. (We use C_a and $\tilde{C}_a$ to distinguish the contours before and after the change of variables.) The conditions (2.25)–(2.27) for $j \neq 2n$ are unchanged for either $\varepsilon_{2n} = +$ or $\varepsilon_{2n} = -$, and for both LHS and RHS. As for the case $\varepsilon_{2n} = -$ and $j = 2n$, the condition is that, in the LHS,

$$\beta_{2n} - \mathrm{i}\lambda \pm \mathrm{i}(n_1\lambda + n_2\pi/\nu + \pi/2) \quad (n_1, n_2 \geqslant 0) \tag{B.9}$$

are above (below) $\tilde{C}_a$; in the RHS,

$$\beta_{2n} \pm \mathrm{i}(n_1\lambda + n_2\pi/\nu + \pi/2) \quad (n_1, n_2 \geqslant 0) \tag{B.10}$$

are above (below) $\tilde{C}_a$. They are not the same, but not contradictory, i.e. no points are required to be on opposite sides of a contour at the same time. Because we know that the integrands are the same, it means that points that appear only in either (B.9) or (B.10) are actually not poles. Therefore, the difference between (B.9) and (B.10) makes no difference in the integrals. As for the case $\varepsilon_{2n} = +$ and $j = 2n$, the conditions (2.26) and (2.27) are unchanged for α_a $(a \neq n)$.

If $\varepsilon_{2n} = +$, the contour for α_n is such that for $j \neq 2n$, in the LHS

$$\beta_j + \mathrm{i}\lambda \pm \mathrm{i}(n_1\lambda + n_2\pi/\nu + \pi/2) \quad (n_1, n_2 \geqslant 0)$$

are above (below) the contour for α_n; in the RHS

$$\beta_j \pm \mathrm{i}(n_1\lambda + n_2\pi/\nu + \pi/2) \quad (n_1, n_2 \geqslant 0)$$

are above (below) the contour for α_n. These two conditions are not contradictory in the same sense as above. For $j = 2n$, the conditions (2.26) and (2.27) are unchanged.

If $\varepsilon_{2n} = +$, the mutual position of α_a and α_n changes from the original one because of the change of variables. However, the resulting positions of α_a and α_n in the LHS and the RHS are identical. Therefore, the integrals are the same.

Proof of (2.6). The factor $\rho(\beta_{2n-1} - \beta_{2n})$ has a zero at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$: we see from (2.12) that

$$\rho(\beta_{2n-1} - \beta_{2n}) = \frac{\nu\mathrm{i}\rho(\pi\mathrm{i})}{\sin\pi\nu}(\beta_{2n} - \beta_{2n-1} - \pi\mathrm{i}) + \cdots. \tag{B.11}$$

The integral may have a pole at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$ because the contour C_a is pinched by the pole of $\varphi(\alpha_a - \beta_{2n-1})$ at $\alpha_a = \beta_{2n-1} + \pi\mathrm{i}/2$ and that of $\varphi(\alpha_a - \beta_{2n})$ at $\alpha_a = \beta_{2n} - \pi\mathrm{i}/2$. We will check if this is indeed a pole, and, if so, compute the residue.

Let us consider the four cases separately.

- Case $(\varepsilon_{2n-1}, \varepsilon_{2n}) = (-, -)$. The pole of $\varphi(\alpha_a - \beta_{2n-1})$ at $\alpha_a = \beta_{2n-1} + \pi\mathrm{i}/2$ is cancelled by the zero of $\sinh\nu(\beta_{2n-1} - \alpha_a + \pi\mathrm{i}/2)$. Therefore, there is no pinching in this case.
- Case $(\varepsilon_{2n-1}, \varepsilon_{2n}) = (+, +)$. If $\bar{a} \neq 2n-1, 2n$, for the same reason, there is no pinching of C_a. Consider the integrals I_i $(i = 1, 2, 3)$ corresponding to the following contours. The integral I_3 has no pinching at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$. Let us show that $I_1 - I_2$ and $I_2 - I_3$ are regular at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$. After integration with respect to $\alpha_1, \ldots, \alpha_{n-2}$, the integral reads as

$$\int \frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\alpha_n}{2\pi\mathrm{i}} A(\alpha_{n-1}, \alpha_n) \prod_{\substack{a=n-1,n \\ j=2n-1,2n}} \varphi(\alpha_a - \beta_j)$$
$$\times \psi(\alpha_{n-1} - \alpha_n) \frac{\sinh\nu(\alpha_n - \beta_{2n-1} + \pi\mathrm{i}/2)\sinh\nu(\beta_{2n} - \alpha_{n-1} + \pi\mathrm{i}/2)}{\sinh\nu(\alpha_{n-1} - \alpha_n - \pi\mathrm{i})}.$$

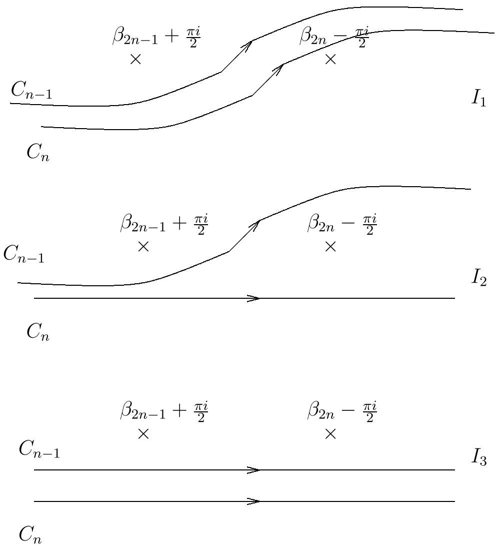


Figure B1. The contours for I_1, I_2 and I_3.

Here, $A(\alpha_{n-1}, \alpha_n)$ is holomorphic and symmetric with respect to α_{n-1} and α_n. Since $\psi(\beta) = -\psi(-\beta)$, we can antisymmetrize the last factor and obtain

$$B(\alpha_{n-1}, \alpha_n; \beta_{2n-1}, \beta_{2n}) = \frac{1}{2}\left\{ \frac{\sinh\nu(\alpha_n - \beta_{2n-1} + \pi\mathrm{i}/2)\sinh\nu(\beta_{2n} - \alpha_{n-1} + \pi\mathrm{i}/2)}{\sinh\nu(\alpha_{n-1} - \alpha_n - \pi\mathrm{i})} - \frac{\sinh\nu(\alpha_{n-1} - \beta_{2n-1} + \pi\mathrm{i}/2)\sinh\nu(\beta_{2n} - \alpha_n + \pi\mathrm{i}/2)}{\sinh\nu(\alpha_n - \alpha_{n-1} - \pi\mathrm{i})} \right\}.$$

The integral $I_1 - I_2$ is equal to the integral over the contour.

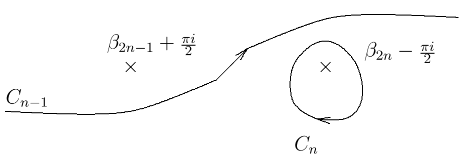


Figure B2. The contour for $I_1 - I_2$.

Taking the residue at $\alpha_n = \beta_{2n} - \pi\mathrm{i}/2$ (the minus sign in front of Res below comes from the clockwise orientation of the integration contour), we get

$$\int \frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}} \left\{-\mathrm{Res}_{\alpha_n=\beta_{2n}-\pi\mathrm{i}/2}\varphi(\alpha_n - \beta_{2n})\right\} A(\alpha_{n-1}, \beta_{2n} - \pi\mathrm{i}/2) \times\varphi(\alpha_{n-1} - \beta_{2n-1})\varphi(\alpha_{n-1} - \beta_{2n})\varphi(\beta_{2n} - \beta_{2n-1} - \pi\mathrm{i}/2) \times\psi(\alpha_{n-1} - \beta_{2n} + \pi\mathrm{i}/2)B(\alpha_{n-1}, \beta_{2n} - \pi\mathrm{i}/2; \beta_{2n-1}, \beta_{2n}).$$

The integrand has no pole at $\alpha_{n-1} = \beta_{2n} - \pi\mathrm{i}/2$ because $\psi(\alpha_{n-1} - \beta_{2n} + \pi\mathrm{i}/2)$ vanishes. The integral has no pole at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$ because $B(\alpha_{n-1}, \beta_{2n} - \pi\mathrm{i}/2; \beta_{2n-1}, \beta_{2n})$ vanishes. Therefore, the integral is regular at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$. By a similar argument we can show that $I_2 - I_3$ is regular at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$.

- Case $(\varepsilon_{2n-1}, \varepsilon_{2n}) = (-, +)$. Taking into account the zero of the factor $\rho(\beta_{2n-1} - \beta_{2n})$ and the pole of the residue $-\mathrm{Res}_{\alpha_n=\beta_{2n}-\pi\mathrm{i}/2}$, both at $\beta_{2n} = \beta_{2n-1} + \pi\mathrm{i}$, we have

$$\frac{c_n}{c_{n-1}}\left\{\rho(\beta_{2n-1} - \beta_{2n})\varphi(\alpha_n - \beta_{2n-1})\left(-\mathrm{Res}_{\alpha_n=\beta_{2n}-\pi\mathrm{i}/2}\varphi(\alpha_n - \beta_{2n})\right)\right.$$

$$
\begin{aligned}
&\times \sinh\nu(\alpha_n-\beta_{2n-1}+\pi \mathrm{i}/2)\\
&\times \prod_{1\leqslant j\leqslant 2n-2}\rho(\beta_j-\beta_{2n-1})\rho(\beta_j-\beta_{2n})\varphi(\alpha_n-\beta_j)\sinh\nu(\alpha_n-\beta_j+\pi \mathrm{i}/2)\\
&\times \prod_{1\leqslant a\leqslant n-1}\varphi(\alpha_a-\beta_{2n-1})\varphi(\alpha_a-\beta_{2n})\psi(\alpha_a-\alpha_n)\\
&\times \left.\frac{\sinh\nu(\beta_{2n-1}-\alpha_a+\pi \mathrm{i}/2)\sinh\nu(\beta_{2n}-\alpha_a+x\pi \mathrm{i}/2)}{\sinh\nu(\alpha_a-\alpha_n-\pi \mathrm{i})}\right\}\Bigg|_{\substack{\alpha_n=\beta_{2n}-\pi \mathrm{i}/2\\ \beta_{2n}=\beta_{2n-1}+\pi \mathrm{i}}}=1\,.
\end{aligned}
$$

Using equations (2.10), (2.12), (2.13), (2.15) and $c_0=1$, we obtain (2.17).

The case $(\varepsilon_{2n-1},\varepsilon_{2n})=(+,-)$ is similar.

Appendix C. One-time integration

In this appendix we show how to reduce the n-fold integral for $G(\beta_1,\ldots,\beta_{2n})$ to an $(n-1)$-fold integral (2.28). We shall follow the method suggested earlier to us by Smirnov. A similar calculation has been published in Nakayashiki's paper [12] where the limiting case $\nu\to 0$ was discussed. Since our working is entirely similar to that in [12], we shall only indicate the necessary steps, omitting further details. In what follows we set $\beta=(\beta_1,\ldots,\beta_{2n})$, $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_{2n})$. We restrict to $\lambda=2\pi$, so that $\psi(\beta)=\sinh\beta$ in the general formula (2.18).

Step 1. Using

$$\prod_{r>s}2\sinh(\alpha_r-\alpha_s)=\det\big(\mathrm{e}^{-(n-2l+1)\alpha_k}\big)_{1\leqslant k,l\leqslant n}$$

we rewrite the main part of $G(\beta)_\varepsilon$ as

$$J(\beta)_\varepsilon=\int\cdots\int\prod_{k=1}^{n}\mathrm{d}\alpha_k\prod_{k,j}\varphi(\alpha_k-\beta_j)\det\big(\mathrm{e}^{-(n-2l+1)\alpha_k}\big)_{1\leqslant k,l\leqslant n}Q(\alpha|\beta)_\varepsilon \tag{C.1}$$

where

$$Q(\alpha|\beta)_\varepsilon=\frac{\prod_{j<\bar{k}}\sinh\nu\big(\alpha_k-\beta_j+\pi \mathrm{i}/2\big)\prod_{j>\bar{k}}\sinh\nu\big(-\alpha_k+\beta_j+\pi \mathrm{i}/2\big)}{\prod_{r<s}\sinh\nu(\alpha_r-\alpha_s-\pi \mathrm{i})}\,.$$

Step 2. In the first column of the determinant, substitute $\mathrm{e}^{-(n-1)\alpha_k}$ by the right-hand side of the identity

$$\mathrm{e}^{-(n-1)\alpha_k}=\frac{\mathrm{i}^{n+1}2^{2n-1}}{\mathrm{e}^{\sum_j\beta_j/2}\sum_j\mathrm{e}^{-\beta_j}}\left(F_+(\alpha_k)-F_-(\alpha_k)+\sum_{l=2}^{n}c_l(\beta)\mathrm{e}^{-(n-2l+1)\alpha_k}\right).$$

Here

$$F_+(\alpha)=\prod_{j=1}^{2n}\sinh\frac{1}{2}\left(\alpha-\beta_j+\frac{\pi \mathrm{i}}{2}\right)\qquad F_-(\alpha)=(-1)^n\prod_{j=1}^{2n}\sinh\frac{1}{2}\left(\alpha-\beta_j-\frac{\pi \mathrm{i}}{2}\right)$$

and $c_l(\beta)$ denotes some function of β_j's. Terms containing $c_l(\beta)$ vanish in the determinant.

Step 3. Expand the determinant at the first column to obtain

$$J(\beta)_\varepsilon = \frac{\mathrm{i}^{n+1}2^{2n-1}}{\mathrm{e}^{\sum_j \beta_j/2}\sum_j \mathrm{e}^{-\beta_j}} \sum_{l=1}^{n} (-1)^{l-1} J_{l,\varepsilon}$$

$$J_{l,\varepsilon} = \int\cdots\int \prod_{k=1}^{n} \mathrm{d}\alpha_k \prod_{k,j} \varphi(\alpha_k - \beta_j)\,(F_+(\alpha_l) - F_-(\alpha_l))\, D_l(\alpha) Q(\alpha|\beta)_\varepsilon$$

where for brevity we set $D_l(\alpha) = \mathrm{D}(\alpha_1, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_n)$.

Step 4. Next we carry out the integral over α_l. For each l, set

$$Q_l(\alpha|\beta)_\varepsilon = \frac{\prod_{j(<\bar{l})} \sinh\nu\big(\alpha_l - \beta_j + \pi\mathrm{i}/2\big) \prod_{j(>\bar{l})} \sinh\nu\big(-\alpha_l + \beta_j + \pi\mathrm{i}/2\big)}{\prod_{r(<l)} \sinh\nu(\alpha_r - \alpha_l - \pi\mathrm{i}) \prod_{r(>l)} \sinh\nu(\alpha_l - \alpha_r - \pi\mathrm{i})}\,.$$

Consider the integrals

$$K^{(\pm)}_{l,\varepsilon} = \pm\int_{C_\pm} H^{(\pm)}_{l,\varepsilon}\,\mathrm{d}\alpha_l$$

$$H^{(\pm)}_{l,\varepsilon} = F_\pm(\alpha_l) \prod_{j=1}^{2n} \varphi(\alpha_l - \beta_j) Q_l(\alpha|\beta)_\varepsilon$$

taken along the contours $C_\pm = \sum_{i=1}^{4} C_{\pm,i}$, respectively, which are shown in figure C1. (To fix the idea we draw the figure assuming the β_j are all real, but the necessary modification in the general situation should be obvious.)

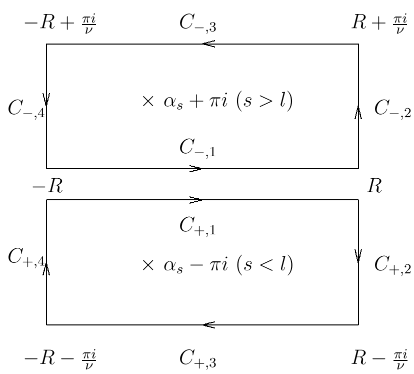


Figure C1. The contour $C_\pm$.

It can be verified that inside the contours the only poles of the integrand $H^{(\pm)}_{l,\varepsilon}$ are $\alpha_l = \alpha_s \mp \pi\mathrm{i}$ for $s < l$ or $s > l$, respectively. Collecting the residues we obtain

$$K^{(+)}_{l,\varepsilon} + K^{(-)}_{l,\varepsilon} = -2\pi\mathrm{i}\left(\sum_{s(<l)} M^{(+)}_{l,s} + \sum_{s(>l)} M^{(-)}_{l,s}\right)$$

$$M^{(\pm)}_{l,s} = \mathrm{Res}_{\alpha_l=\alpha_s\mp\pi\mathrm{i}} F_\pm(\alpha_l) \prod_j \varphi(\alpha_l - \beta_j)\, Q_l(\alpha|\beta)_\varepsilon\, \mathrm{d}\alpha_l\,.$$

One can show that, upon integration by the other variables and summing over l, these terms cancel with each other. More precisely, set

$$Q'_{l,s}(\alpha|\beta)_\varepsilon = \frac{\prod_{j(<\bar{l})} \sinh\nu\big(\alpha_l - \beta_j + \pi\mathrm{i}/2\big) \prod_{j(>\bar{l})} \sinh\nu\big(-\alpha_l + \beta_j + \pi\mathrm{i}/2\big)}{\prod_{\substack{r(<l)\\ r\neq s}} \sinh\nu(\alpha_r - \alpha_l - \pi\mathrm{i}) \prod_{\substack{r(>l)\\ r\neq s}} \sinh\nu(\alpha_l - \alpha_r - \pi\mathrm{i})}\,.$$

Then we have, for any pair $r < s$,

$$\int d\alpha_r\, M^{(+)}_{sr} D_s(\alpha) Q'_{rs}(\alpha|\beta)_\varepsilon + (-1)^{r-s} \int d\alpha_s\, M^{(-)}_{rs} D_r(\alpha) Q'_{sr}(\alpha|\beta)_\varepsilon = 0\,.$$

This can be seen by changing the variable $\alpha_r \to \alpha_s + \pi\mathrm{i}$.

Step 5. From the transformation properties of $\varphi(\beta)$ it follows that

$$H^{(\pm)}_{l,\varepsilon}\big|_{\alpha_l\to\alpha_l\mp\pi\mathrm{i}/\nu} = H^{(\mp)}_{l,\varepsilon}$$

which implies that the integrals corresponding to $C_{\pm,3}$ give the same results as for $C_{\mp,1}$.

As $R \to \infty$, the integrals along $C_{\pm,2,4}$ are calculated from the following asymptotics of the integrand as $\alpha_l \to \pm\infty$:

$$\varphi(\alpha_l - \beta_j) \sim 2\exp\left(\mp\frac{1+\nu}{2}(\alpha_l - \beta_j)\right)$$

$$F_\sigma(\alpha_l)\prod_{j=1}^{2n}\varphi(\alpha_l-\beta_j) \sim \mathrm{i}^n \exp\left(\mp\nu\left(n\alpha_l - \tfrac{1}{2}\sum_{j=1}^{2n}\beta_j\right)\right)$$

$$Q_l(\alpha|\beta)_\varepsilon \sim \frac{(-1)^{l+\bar{l}+1}}{2^n}\exp\big(\pm\nu(n\alpha_l + A_l)\big) \times (\pm 1)^n$$

with

$$A_l = \sum_{k\neq l}\alpha_k - \sum_{j\neq\bar{l}}\beta_j + \pi\mathrm{i}\big(\bar{l} - 2l + \tfrac{1}{2}\big)\,.$$

From the last two steps we find that

$$\int_{-R}^{R}\big(H^{(+)}_{l,\varepsilon} - H^{(-)}_{l,\varepsilon}\big)\, d\alpha_l = \frac{\pi\mathrm{i}}{\nu}\frac{(-1)^{l+\bar{l}+1}\mathrm{i}^n}{2^n}\big(\mathrm{e}^{\nu\tilde{A}_l} - \mathrm{e}^{-\nu\tilde{A}_l}\big) + \mathcal{R}$$

where

$$\tilde{A}_l = A_l + \tfrac{1}{2}\sum_{j=1}^{2n}\beta_j$$

and $\mathcal{R}$ signifies a term which vanishes when $R \to \infty$. Hence we arrive at the result (2.28) stated in the beginning.

Appendix D. Derivation of the nearest-neighbour correlator

Here we derive the formula (4.3). We start from (4.2) and consider the specialization

$$G = G(\beta+\pi\mathrm{i}, \beta+\pi\mathrm{i}, \beta, \beta)_{++--}\,.$$

Proposition D.1.

$$G + \frac{1}{2} = \frac{1}{2\pi\nu^2}\int_{-\infty}^{\infty}\frac{\mathrm{e}^\alpha}{\cosh^2\alpha}\frac{\varphi'(\alpha)}{\varphi(\alpha)}\,(1 - \cos\pi\nu\cosh 2\nu\alpha)\, d\alpha\,. \tag{D.1}$$

Proof. First let $(\beta_1, \ldots, \beta_4) = (\beta + \pi\mathrm{i}, \beta + \pi\mathrm{i}, \beta + \varepsilon, \beta + \varepsilon)$. Then (4.2) becomes

$$\begin{aligned}G(\beta + \pi\mathrm{i}, \beta + \pi\mathrm{i}, \beta + \varepsilon, \beta + \varepsilon)_{++--} &= \frac{-1}{4\pi\nu^2}\frac{\mathrm{e}^{-\beta-\varepsilon}}{1-\mathrm{e}^{-\varepsilon}}\frac{\rho(\pi\mathrm{i}-\varepsilon)^4}{\rho(\pi\mathrm{i})^4}\\ &\times\int \mathrm{d}\alpha\, \mathrm{e}^{\alpha}\varphi(\alpha-\beta-\pi\mathrm{i})^2\varphi(\alpha-\beta-\varepsilon)^2\sinh^2\nu\left(\alpha-\beta-\varepsilon-\frac{\pi\mathrm{i}}{2}\right)\\ &\times\left[\sinh\nu\left(\alpha-\beta-\frac{3\pi\mathrm{i}}{2}\right)\sinh\nu\left(\alpha-\beta-\varepsilon-\frac{3\pi\mathrm{i}}{2}\right)\right.\\ &\left.+\sinh\nu\left(\alpha-\beta-\frac{\pi\mathrm{i}}{2}\right)\sinh\nu\left(\alpha-\beta-\varepsilon-\frac{\pi\mathrm{i}}{2}\right)\right].\end{aligned}$$

Since $\alpha = \beta+\varepsilon+\pi\mathrm{i}/2$ is not a pole, the contour can be taken as $\pi/2 < \mathrm{Im}(\alpha-\beta) < 3\pi/2$. Changing the variable $\alpha \to \alpha+\beta+\varepsilon+\pi\mathrm{i}$ and using $\varphi(\alpha+\pi\mathrm{i})\varphi(\alpha) = -\mathrm{i}/\cosh\alpha\sinh\nu(\alpha+\pi\mathrm{i}/2)$, we find

$$\begin{aligned}G = \lim_{\varepsilon\to 0}\frac{1}{4\pi\nu^2}\frac{1}{1-\mathrm{e}^{-\varepsilon}}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\mathrm{e}^{\alpha}\frac{\varphi(\alpha+\varepsilon)^2}{\varphi(\alpha)^2}\frac{-1}{\cosh^2\alpha}&\left[\sinh\nu\left(\alpha+\frac{\pi\mathrm{i}}{2}+\varepsilon\right)\sinh\nu\left(\alpha+\frac{\pi\mathrm{i}}{2}\right)\right.\\ &\left.+\sinh\nu\left(\alpha-\frac{\pi\mathrm{i}}{2}+\varepsilon\right)\sinh\nu\left(\alpha-\frac{\pi\mathrm{i}}{2}\right)\right].\end{aligned}$$

Noting

$$0 = \int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\alpha}}{\cosh^2\alpha}\left[\sinh^2\nu\left(\alpha+\frac{\pi\mathrm{i}}{2}\right)+\sinh^2\nu\left(\alpha-\frac{\pi\mathrm{i}}{2}\right)\right]$$

and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}G &= \frac{1}{2\pi\nu^2}\frac{\partial}{\partial\varepsilon}\left(\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\mathrm{e}^{\alpha}\frac{\varphi(\alpha+\varepsilon)^2}{\varphi(\alpha)^2}\frac{-1}{\cosh^2\alpha}\,\mathrm{Re}\sinh\nu\left(\alpha+\frac{\pi\mathrm{i}}{2}+\varepsilon\right)\sinh\nu\left(\alpha+\frac{\pi\mathrm{i}}{2}\right)\right)\Bigg|_{\varepsilon=0}\\ &= I_1 + I_2\end{aligned}$$

where I_1 is given by the right-hand side of (D.1) and

$$I_2 = \frac{-1}{4\pi\nu}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\alpha}}{\cosh^2\alpha}\;\mathrm{Re}\sinh 2\nu\left(\alpha+\frac{\pi\mathrm{i}}{2}\right).$$

Using

$$\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\lambda\alpha}}{\cosh^2\alpha} = \frac{\pi\lambda}{\sin\pi\lambda/2} \tag{D.2}$$

we find

$$I_2 = -\tfrac{1}{2}$$

thereby completing the proof of the lemma. □

Proposition D.2.

$$G = J_1 + J_2 - \tfrac{1}{2}$$

$$J_1 = \frac{1}{\pi^2}\int_0^{\infty} t\,\mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t}\left(\frac{1}{\sin\pi\nu}\,\mathrm{Im}\,\frac{1}{\cosh\nu(t+\pi\mathrm{i})}+\frac{\sinh\nu t}{\cosh^2\nu t}\right) \tag{D.3}$$

$$J_2 = \frac{1}{\pi\sin\pi\nu}\int_0^{\infty}\mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t}\left(\mathrm{Re}\,\frac{1}{\cosh\nu(t+\pi\mathrm{i})}-\frac{\cos\pi\nu}{\cosh\nu t}\right).$$

Proof. Substituting the integral formula for $\log\varphi(\alpha)$ into I_1 above, we obtain

$$G+\tfrac{1}{2}=\frac{1}{\pi^2\nu}\int_0^\infty \mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t\,\sinh 2\nu t}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{\mathrm{e}^\alpha}{\cosh^2\alpha}\sin\left(\frac{2\nu t}{\pi}\alpha\right)(\cosh 2\nu\alpha\cos\pi\nu-1)\,.$$

By integrating over α using (D.2), the right-hand side becomes

$$\frac{1}{\pi^2}\int_0^\infty \mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t\,\sinh 2\nu t}\left(\cos\pi\nu\left(\frac{t-\pi\mathrm{i}}{\cosh\nu(t-\pi\mathrm{i})}+\frac{t+\pi\mathrm{i}}{\cosh\nu(t+\pi\mathrm{i})}\right)-\frac{2t}{\cosh\nu t}\right).$$

After a little algebra we obtain (D.3). □

Proposition D.3. We have

$$G-\tfrac{1}{2}=J_1+J_2-1=\frac{1}{\pi^2\sin\pi\nu}\frac{\partial}{\partial\nu}\left(\sin\pi\nu\int_0^\infty \mathrm{d}t\,\frac{\sinh(1-\nu)t}{\sinh t\,\cosh\nu t}\right). \qquad \text{(D.4)}$$

Proof. Consider the integral

$$\int_C \mathrm{d}t\,\frac{\sinh(1+\nu)(t-\pi\mathrm{i})}{\sinh(t-\pi\mathrm{i})}\,\frac{t-\pi\mathrm{i}}{\cosh\nu t}=0$$

where the contour C is as shown in figure D1.

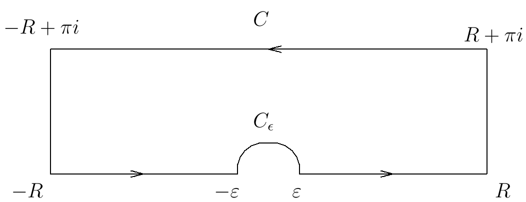


Figure D1. The contour C and the semicircle C_ε.

Taking the imaginary part, we obtain

$$\begin{aligned}
&\int_{-R}^{R} t\,\mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t}\ \mathrm{Im}\left(\frac{1}{\cosh\nu(t+\pi\mathrm{i})}\right)\\
&\qquad=\mathrm{Im}\left(\int_{-R}^{-\varepsilon}+\int_{C_\varepsilon}+\int_{\varepsilon}^{R}\right)\mathrm{d}t\left(\frac{\sinh(1+\nu)(t-\pi\mathrm{i})}{\sinh(t-\pi\mathrm{i})}\,\frac{t-\pi\mathrm{i}}{\cosh\nu t}\right)\\
&\qquad\quad+\int_0^\pi \mathrm{d}t\ \mathrm{Re}\left(\frac{\sinh(1+\nu)(R+t\mathrm{i}-\pi\mathrm{i})}{\sinh(R+t\mathrm{i}-\pi\mathrm{i})}\,\frac{R+t\mathrm{i}-\pi\mathrm{i}}{\cosh\nu(R+t\mathrm{i})}\right.\\
&\qquad\quad\left.-\frac{\sinh(1+\nu)(-R+t\mathrm{i}-\pi\mathrm{i})}{\sinh(-R+t\mathrm{i}-\pi\mathrm{i})}\,\frac{-R+t\mathrm{i}-\pi\mathrm{i}}{\cosh\nu(-R+t\mathrm{i})}\right).
\end{aligned}$$

As $\varepsilon\to 0$, the integral over the semicircle C_ε gives $\pi^2\sin\pi\nu$, and as $R\to\infty$, the last term behaves like

$$4\pi R\cos\pi\nu-2\pi^2\sin\pi\nu+\mathrm{O}(R^{-1})\,.$$

Since

$$\mathrm{Im}\left(\frac{\sinh(1+\nu)(t-\pi\mathrm{i})}{\sinh(t-\pi\mathrm{i})}\,\frac{t-\pi\mathrm{i}}{\cosh\nu t}\right)=-\frac{t\cosh(1+\nu)t\sin\pi\nu}{\sinh t\cosh\nu t}-\frac{\pi\sinh(1+\nu)t\cos\pi\nu}{\sinh t\cosh\nu t}$$

and

$$\frac{\sinh(1+\nu)t\sinh\nu t}{\sinh t\cosh^2\nu t}-\frac{\cosh(1+\nu)t}{\sinh t\cosh^2\nu t}=-\frac{\cosh t}{\sinh t\cosh\nu t}$$

we have

$$\int_{-R}^{R} t\,\mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t}\left(\frac{\sinh\nu t}{\cosh^2\nu t}+\frac{1}{\sin\pi\nu}\,\mathrm{Im}\,\frac{1}{\cosh\nu(t+\pi\mathrm{i})}\right)$$
$$= -\int_{-R}^{R} t\,\mathrm{d}t\,\frac{\cosh t}{\sinh t\cosh^2\nu t}-\pi\cot\pi\nu\int_{-R}^{R}\frac{\sinh(1+\nu)t}{\sinh t\cosh\nu t}\,\mathrm{d}t$$
$$+4\pi R\cot\pi\nu-\pi^2+\mathrm{O}(R^{-1})\,.$$

Noting that

$$\lim_{R\to\infty}\left(4\pi R\cot\pi\nu-\pi\cot\pi\nu\int_{-R}^{R}\mathrm{d}t\,\frac{\sinh(1+\nu)t}{\sinh t\cosh\nu t}\right)=2\pi\cot\pi\nu\int_0^\infty\mathrm{d}t\,\frac{\sinh(1-\nu)t}{\sinh t\cosh\nu t}$$

we have

$$J_1=-\frac{1}{2}-\frac{1}{\pi^2}\int_0^\infty t\,\mathrm{d}t\,\frac{\cosh t}{\sinh t\cosh^2\nu t}+\frac{\cot\pi\nu}{\pi}\int_0^\infty\mathrm{d}t\,\frac{\sinh(1-\nu)t}{\sinh t\cosh\nu t}\,.$$

By a similar calculation, starting from

$$\int_C\mathrm{d}t\,\frac{\sinh(1+\nu)(t-\pi\mathrm{i})}{\sinh(t-\pi\mathrm{i})}\,\frac{1}{\cosh\nu t}=0$$

we obtain $J_2=\frac{3}{2}$. Collecting these terms, we obtain (D.4). □

Appendix E. Integrals related to the case $\nu=\frac{1}{2}$

Here we supply proofs to the formulae presented in section 5. We retain the notation used there.

First let us evaluate the integrals I_{ki} (5.2) and J_{ki} (E.5). Set $B_j=\mathrm{e}^{\beta_j}$, and define

$$\begin{aligned}F_{ji}&=\frac{1}{\prod_{k=r,\,k\neq j}^{i}(B_j-B_k)\prod_{k=i}^{s}(B_j+B_k)}\qquad(r\leqslant j\leqslant i\leqslant s)\\ G_{ji}&=\frac{(-1)^n}{\prod_{k=r}^{i}(B_j+B_k)\prod_{k=i,\,k\neq j}^{s}(B_j-B_k)}\qquad(r\leqslant i\leqslant j\leqslant s)\,.\end{aligned}\tag{E.1}$$

Proposition E.1. For $1\leqslant k\leqslant n$ and $r\leqslant i\leqslant s$ we have

$$I_{ki}=\frac{\mathrm{i}^{s+1-i}}{\pi}B_i^{1/2}\left(\prod_{j=r}^{s}B_j\right)^{1/2}\left(\sum_{j=r}^{i}(-B_j)^{k-1}\beta_jF_{ji}+\sum_{j=i}^{s}B_j^{k-1}(\beta_j+\pi\mathrm{i})G_{ji}\right)\tag{E.2}$$

$$J_{ki}=2\mathrm{i}^{s-i}B_i^{1/2}\left(\prod_{j=r}^{s}B_j\right)^{1/2}\sum_{j=r}^{i}(-B_j)^{k-1}F_{ji}\tag{E.3}$$

$$=-2\mathrm{i}^{s-i}B_i^{1/2}\left(\prod_{j=r}^{s}B_j\right)^{1/2}\sum_{j=i}^{s}B_j^{k-1}G_{ji}\,.\tag{E.4}$$

Proof. Changing the integration variable to $A=\mathrm{e}^{\alpha}$ we have

$$I_{ki}=\frac{\mathrm{i}^{2r-s+1-i}}{\pi}B_i^{1/2}\left(\prod_{j=r}^{s}B_j\right)^{1/2}\int_0^\infty\omega_{ki}$$

where we have set

$$\omega_{ki} = \frac{A^{k-1}\,\mathrm{d}A}{\prod_{j=r}^{i}(A+B_j)\prod_{j=i}^{s}(A-B_j+\mathrm{i}\,0)}\,.$$

To see (E.2) it suffices to show that

$$\int_0^\infty \omega_{ki} = (-1)^{n+1}\left(\sum_{j=r}^{i}(-B_j)^{k-1}\beta_j F_{ji} + \sum_{j=i}^{s} B_j^{k-1}(\beta_j+\pi\mathrm{i})G_{ji}\right).$$

This follows from integration of $\omega_{ki}\log(-A)$ along the contour shown in figure E1.

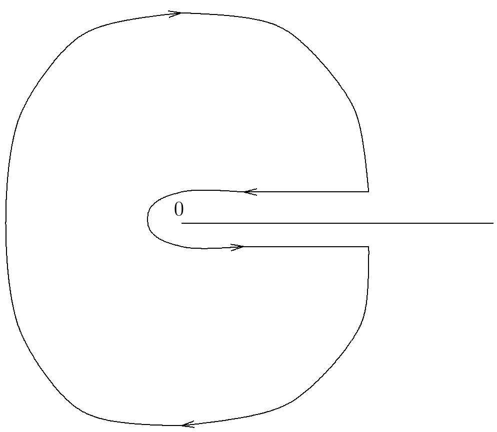


Figure E1. The contour for the residue calculus.

The formula (E.4) is a direct consequence of (E.2). Counting the sum of the residues of ω_{ki} we find

$$0 = \sum_{j=r}^{i}(-B_j)^{k-1}F_{ji} + \sum_{j=i}^{s} B_j^{k-1}G_{ji}\,.$$

This shows the equality of (E.3) and (E.4). □

Define

$$J_{ki} = I_{ki} + (-1)^{s+i}\bar{I}_{ki} \tag{E.5}$$

and denote by I_i and J_i the column vectors

$$I_i = {}^{\mathrm{t}}(I_{1i},\ldots,I_{ni}) \qquad J_i = {}^{\mathrm{t}}(J_{1i},\ldots,J_{ni})\,.$$

Proposition E.2.

$$1 = \prod_{r\leqslant j<k\leqslant s} 2\mathrm{i}\cosh\frac{\beta_j-\beta_k}{2}\,\det(J_r, J_{r+1},\ldots,J_s)\,. \tag{E.6}$$

Proof. To see this, we write down the known equality

$$1 = \langle 1\rangle = \sum_{\varepsilon_r,\cdots,\varepsilon_s}\langle E^{(r)}_{\varepsilon_r\varepsilon_r}\cdots E^{(s)}_{\varepsilon_s\varepsilon_s}\rangle(\beta_r,\ldots,\beta_s)\,. \tag{E.7}$$

For convenience let us introduce the symbols $I_i(\pm)$, $\bar{I}_i(\pm)$ by setting $I_i(+) = I_i$, $\bar{I}_i(+) = \bar{I}_i$ and $I_i(-)$, $\bar{I}_i(-) =$ the empty symbol. Then (E.7) can be written as

$$1 = \prod_{r\leqslant j<k\leqslant s} 2\mathrm{i}\cosh\frac{\beta_j-\beta_k}{2}\sum_{\varepsilon_r,\ldots,\varepsilon_s}\det\big(I_r(-\varepsilon_r),\ldots,I_s(-\varepsilon_s),\bar{I}_s(\varepsilon_s),\ldots,\bar{I}_r(\varepsilon_r)\big)\,.$$

Taking the sum over $\varepsilon_s, \varepsilon_{s-1}, \ldots$ successively and keeping track of the signs, we find that the last sum becomes a single determinant $\det(J_r, \ldots, J_s)$. □

Proposition E.3. If $r \leqslant m, l \leqslant s$ then

$$\langle \psi_m \psi_l^* \rangle = D^{-1} \det(J_r, \ldots, J_{l-1}, I_m, J_{l+1}, \ldots, J_s) \tag{E.8}$$

where $D = \det(J_r, J_{r+1}, \ldots, J_s)$.

Proof. This can be verified in a similar way as in the proof of proposition E.2. As an example, let us take $r = 1, s = 4, m = 1, l = 3$:

$$\begin{aligned}\langle \psi_1 \psi_3^* \rangle &= \langle \sigma_1^- \sigma_2^z \sigma_3^+ 1 \rangle \\ &= \sum_{\varepsilon_2, \varepsilon_4} \langle E_{-+}^{(1)} \left(\varepsilon_2 E_{\varepsilon_2\varepsilon_2}^{(2)}\right) E_{+-}^{(3)} E_{\varepsilon_4\varepsilon_4}^{(4)} \rangle .\end{aligned}$$

Using equations (5.1) and (E.6), in the same notation as in proposition E.2, we have

$$D\langle \psi_1 \psi_3^* \rangle = \sum_{\varepsilon_2, \varepsilon_4} \varepsilon_2 \det\big(I_1(+), I_2(-\varepsilon_2), I_3(-), I_4(-\varepsilon_4)\bar{I}_4(\varepsilon_4), \bar{I}_3(-), \bar{I}_2(\varepsilon_2), \bar{I}_1(+)\big) .$$

The sum over ε_4 gives J_4, Summing further over ε_2 we obtain

$$\begin{aligned}\det(I_1, J_4, \bar{I}_2, \bar{I}_1) - \det(I_1, I_2, J_4, \bar{I}_1) &= \det(I_1, -J_2, J_4, \bar{I}_1 \\ &= \det(J_1, J_2, I_1, J_4)\end{aligned}$$

where we used $J_2 = I_2 + \bar{I}_2$ and $J_1 = I_1 - \bar{I}_1$.

In general, consider the case $m < l$. We have

$$D\langle \psi_m \psi_l^* \rangle = \sum_{\varepsilon_r, \ldots, \varepsilon_s} \varepsilon_{m+1} \cdots \varepsilon_{l-1} \det I(\varepsilon)$$

where $I(\varepsilon)$ denotes the matrix consisting of the following array of column vectors:

$$\begin{aligned}&I_r(-\varepsilon_r), \ldots, I_{m-1}(-\varepsilon_{m-1}), I_m(+), I_{m+1}(-\varepsilon_{m+1}), \ldots, I_{l-1}(-\varepsilon_{l-1}) \\ &I_l(-), I_{l+1}(-\varepsilon_{l+1}), \ldots, I_s(-\varepsilon_s), \bar{I}_s(\varepsilon_s), \ldots, \bar{I}_{l+1}(\varepsilon_{l+1}) \\ &\bar{I}_l(-), \bar{I}_{l-1}(\varepsilon_{l-1}), \ldots, \bar{I}_{m+1}(\varepsilon_{m+1}), \bar{I}_m(+), \bar{I}_{m-1}(\varepsilon_{m-1}), \ldots, \bar{I}_r(\varepsilon_r) .\end{aligned}$$

Summing over $\varepsilon_s, \varepsilon_{s-1}, \ldots$ we see that the sum combines into a single determinant

$$\begin{aligned}&\det(J_r, \ldots, J_{m-1}, I_m, -J_{m+1}, \ldots, -J_{l-1}, J_{l+1}, \ldots, J_s, \bar{I}_m) \\ &\quad = \det(J_r, \ldots, J_{m-1}, (-1)^{s-m}\bar{I}_m, J_{m+1}, \ldots, J_{l-1}, I_m, J_{l+1}, \ldots, J_s) .\end{aligned}$$

This shows (E.8). The other cases are similar. □

Arguing in a similar manner, one can show in general the following.

Proposition E.4. Suppose $m_1 < \cdots < m_k$, $l_1 < \cdots < l_k$, and let $r \leqslant \min(m_1, l_1)$, $\max(m_k, l_k) \leqslant s$. Then

$$\langle \psi_{m_1} \cdots \psi_{m_k} \psi_{l_k}^* \cdots \psi_{l_1}^* \rangle = D^{-1} \det\left(J_r, \ldots, J_{l_1-1}, I_{m_1}, J_{l_1+1}, \ldots, J_{l_k-1}, I_{m_k}, J_{l_k+1}, \ldots, J_s\right)$$

where $D = \det(J_r, J_{r+1}, \ldots, J_s)$. In the right-hand side I_{m_j} is placed at the l_jth slot.

We omit the details.

Proof of proposition 5.2. Consider the matrix $X = (J_r, J_{r+1}, \ldots, J_s)$, and set $K_m = X^{-1} I_m$. Then proposition E.4 states that

$$\langle \psi_{m_1} \cdots \psi_{m_k} \psi_{l_k}^* \cdots \psi_{l_1}^* \rangle = \det(e_1, \ldots, K_{m_1}, \ldots, K_{m_k}, \ldots, e_n)$$

where the $e_j = (\delta_{ji})_{1\leqslant i\leqslant n}$ denote the unit vectors. It is clear that the right-hand side is

$$\det((K_{m_j})_{l_i})_{1\leqslant j,i\leqslant k}\,.$$

and that $\langle\psi_m\psi_l^*\rangle = (K_m)_l$. The proposition follows from this observation. □

Proof of proposition 5.3. The formula (5.5) is already known. Let us show (5.4) by taking $r = m$ and $s = l$ in (E.8). We wish to compute

$$\det\bigl(J_m, J_{m+1}, \ldots, J_{l-1}, I_m - \tfrac{1}{2}J_m\bigr)\,.$$

Substituting

$$\begin{aligned} I_{km} - \tfrac{1}{2}J_{km} &= \frac{\mathrm{i}^n}{\pi}B_m^{1/2}\Biggl(\prod_{j=m}^{l}B_j\Biggr)^{1/2}\Biggl((-B_m)^{k-1}\beta_m F_{mm} + \sum_{j=m}^{l}B_j^{k-1}\beta_j G_{jm}\Biggr) \\ &= \frac{\mathrm{i}^n}{\pi}B_m^{1/2}\Biggl(\prod_{j=m}^{l}B_j\Biggr)^{1/2}\sum_{j=m}^{l}B_j^{k-1}(\beta_j-\beta_m)G_{jm} \end{aligned}$$

and using (E.3), we have the following expression for $D\langle\psi_m\psi_l^*\rangle$:

$$\frac{\mathrm{i}^n}{\pi}B_m^{1/2}\Biggl(\prod_{j=m}^{l}B_j\Biggr)^{n/2}\sum_{j=m}^{l}(\beta_j-\beta_m)G_{jm}\det X_j Y Z\,.$$

Here X_j, Y, Z are the following matrices:

$$X_j = \begin{pmatrix} 1 & \ldots & 1 & 1 \\ -B_m & \ldots & -B_{l-1} & B_j \\ \vdots & & \vdots & \vdots \\ (-B_m)^{n-1} & \ldots & (-B_{l-1})^{n-1} & B_j^{n-1} \end{pmatrix}$$

$$Y = \begin{pmatrix} F_{mm} & \ldots & F_{m\,l-1} & 0 \\ 0 & \ddots & \vdots & 0 \\ 0 & \ldots & F_{l-1\,l-1} & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

$$Z = \mathrm{diag}\bigl(2\mathrm{i}^{\,l-m}B_m^{1/2}, \ldots, 2\mathrm{i}B_{l-1}^{1/2}, 1\bigr)\,.$$

The matrix Y is upper triangular with diagonal entries F_{kk} $(m \leqslant k \leqslant l-1)$ and 1. It is therefore straightforward to compute the determinant. Inserting the expressions for F_{ji} and G_{ji} in (E.1) we obtain the formula (5.4).

The case of $\langle\psi_m^*\psi_l\rangle$ can be shown similarly, using (E.4). □

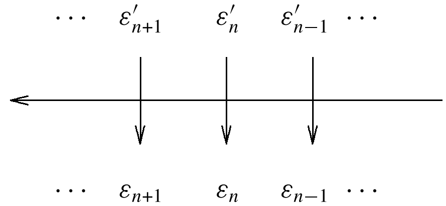
Appendix F. Inhomogeneous Ising model

In this section, we compute the correlation functions of the inhomogeneous Ising model at the critical temperature. We give an explicit formula for the vacuum expectation values $\langle\mathrm{vac}|\psi_m^*\psi_n|\mathrm{vac}\rangle$ where ψ_n^*, ψ_n $(n \in \mathbb{Z})$ are the free fermions diagonalizing the transfer matrix $T(u)$ of the critical Ising model. The general correlation functions are given by the Pfaffians of these matrix elements. In [21] the correlation functions for the critical Ising model were given. We have not checked the equivalence of our result to theirs except for some simple cases.

F.1. Completely inhomogeneous Hamiltonian

Consider the transfer matrix of a completely inhomogeneous six-vertex model in the infinite volume:

$T(u)^{\{\varepsilon'_n\}}_{\{\varepsilon_n\}} =$



$$= \sum_{\{\tau_n\}} \prod_n \bar{R}^{\varepsilon'_n \tau_{n-1}}_{\varepsilon_n \tau_n}(\beta_n + u)\,.$$

The horizontal line carries the spectral parameter 0 and the vertical lines carry the spectral parameters $\beta_n + u$. We assume that $\beta_n = 0$ if $|n| \gg 0$. The Boltzmann weights $\bar{R}^{\varepsilon'_1,\varepsilon'_2}_{\varepsilon_1,\varepsilon_2}(\beta)$ are given by (2.3) with $\nu = \frac{1}{2}$. This is the choice in which the six-vertex model is equivalent to the critical Ising model (see e.g. [3]).

Let S be the shift operator

$$S^{\{\varepsilon'_n\}}_{\{\varepsilon_n\}} = \prod_n \delta_{\varepsilon_{n+1}\varepsilon'_n}\,.$$

Then we have

$$\begin{aligned} S^{-1}T(u) &= \sum_{\{\tau_n\}} \prod_n \check{R}^{\varepsilon'_n \tau_{n-1}}_{\tau_n \varepsilon_{n-1}}(\beta_n + u) \\ &= \cdots \check{R}_{n+1\,n}(\beta_{n+1} + u)\check{R}_{n\,n-1}(\beta_n + u)\cdots \end{aligned} \tag{F.1}$$

where

$$\check{R}(\beta) = \begin{pmatrix} 1 & & & \\ & \bar{c}(\beta) & \bar{b}(\beta) & \\ & \bar{b}(\beta) & \bar{c}(\beta) & \\ & & & 1 \end{pmatrix}$$

and

$$\bar{b}(\beta) = \frac{1-\zeta^2}{\mathrm{i}(1+\zeta^2)} \qquad \bar{c}(\beta) = \frac{2\zeta}{1+\zeta^2} \qquad \zeta = \mathrm{e}^{-\beta/2}\,.$$

The matrix $\check{R}(\beta)$ can be put in the form

$$\check{R}(\beta) = \mathrm{e}^{\gamma X} \qquad X = \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+$$

where $\sigma^{\pm} = (\sigma^x \pm \mathrm{i}\sigma^y)/2$ and $\gamma = \gamma(\beta)$ is related to β by

$$\begin{aligned} \mathrm{e}^{-\gamma} &= \frac{1 + \mathrm{i}\,\sinh\beta/2}{\cosh\beta/2} = \frac{2\zeta}{1+\zeta^2} + \frac{\mathrm{i}(1-\zeta^2)}{1+\zeta^2} \\ \mathrm{e}^{-\beta/2} &= \frac{1 - \mathrm{i}\sinh\gamma}{\cosh\gamma} = \zeta\,. \end{aligned}$$

As $-\mathrm{i}\beta$ increases from 0 to π, γ increases monotonically from 0 to ∞. We write $\gamma_n = \gamma(\beta_n)$, $C_n = \cosh\gamma_n$ and $S_n = \sinh\gamma_n$. In what follows, we assume that $S_n < 1$.

F.2. Jordan–Wigner transformation

As usual, we define the Jordan–Wigner transformation

$$\psi_n^* = \sigma_n^+ \prod_{m<n} \sigma_m^z \qquad \psi_n = \sigma_n^- \prod_{m<n} \sigma_m^z .$$

Note that the operators ψ_n^* and ψ_n satisfy the canonical anti-commutation relation

$$[\psi_m^*, \psi_n^*]_+ = [\psi_m, \psi_n]_+ = 0 \qquad [\psi_m^*, \psi_n]_+ = \delta_{m,n} . \tag{F.2}$$

For $m, n \in \mathbb{Z}$ such that $m > n$, we set

$$H_{mn} = \psi_n \psi_m^* + (-1)^{m-n+1} \psi_m \psi_n^* .$$

Note that

$$X_{n\,n-1} = H_{n\,n-1} \qquad [H_{n\,n-1}, H_{m,n}] = H_{m\,n-1} \qquad [H_{n\,n-1}, H_{m\,n-1}] = H_{m\,n} . \tag{F.3}$$

We have

Proposition F.1.

$$T(0)^{-1} T(u) = 1 + \frac{\mathrm{i}u}{2} \mathcal{H} + \mathrm{O}(u^2) \tag{F.4}$$

where

$$\mathcal{H} = \sum_{m>n} (-1)^{m-n} C_m S_{m-1} \cdots S_{n+1} C_n H_{mn} .$$

Proof. Let us use $\equiv$ to mean an equality modulo u^2. Using (F.1), we have

$$\begin{aligned} T(0)^{-1} T(u) - 1 &\equiv \sum_n \cdots \check{R}_{n-1\,n-2}(\beta_{n-1})^{-1} \big(\check{R}_{n\,n-1}(\beta_n)^{-1} \check{R}_{n\,n-1}(\beta_n + u) - 1 \big) \\ &\quad \times \check{R}_{n-1\,n-2}(\beta_{n-1}) \cdots \\ &\equiv \frac{-\mathrm{i}u}{2} \sum_n \cdots \mathrm{Ad}\, \check{R}_{n-1\,n-2}(\beta_{n-1})^{-1} C_n H_{n\,n-1} . \end{aligned}$$

Since $\check{R}_{k\,k-1}(\beta)^{-1} = \mathrm{e}^{-\gamma_k H_{k\,k-1}}$, the proposition follows from (F.3). □

If we fix β_n's, the transfer matrices $T(u)$ commute with each other for different values of u. Therefore $\mathcal{H}$ also commutes with $T(u)$, and they can be diagonalized simultaneously.

F.3. Diagonalization

In order to diagonalize the Hamiltonian $\mathcal{H}$ we set

$$\begin{aligned} \phi(\theta) &= \sum_n C_n \, \mathrm{e}^{\mathrm{i}n\theta} \prod_{j \leqslant n-1} (1 + S_j \, \mathrm{e}^{-\mathrm{i}\theta}) \prod_{j \geqslant n+1} (1 - S_j \, \mathrm{e}^{\mathrm{i}\theta}) \psi_n \\ \phi^*(\theta) &= \sum_n C_n \, \mathrm{e}^{-\mathrm{i}n\theta} \prod_{j \leqslant n-1} (1 - S_j \, \mathrm{e}^{\mathrm{i}\theta}) \prod_{j \geqslant n+1} (1 + S_j \, \mathrm{e}^{-\mathrm{i}\theta}) \psi_n^* . \end{aligned} \tag{F.5}$$

Then we have

Proposition F.2.

$$[\mathcal{H}, \phi(\theta)] = -(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \phi(\theta) \tag{F.6}$$

$$[\mathcal{H}, \phi^*(\theta)] = (\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \phi^*(\theta) \tag{F.7}$$

$$[\phi^*(\theta_1), \phi(\theta_2)]_+ = \prod_j (1 + S_j \, \mathrm{e}^{-\mathrm{i}\theta_1})(1 - S_j \, \mathrm{e}^{\mathrm{i}\theta_1}) \sum_k \mathrm{e}^{\mathrm{i}k(\theta_1 - \theta_2)} . \tag{F.8}$$

Proof. Set $z = e^{i\theta}$, and write

$$\phi(\theta) = \sum_{n\in\mathbb{Z}} x_n \psi_n$$

$$x_n = C_n z^n \prod_{j\leqslant n-1} (1 + S_j z^{-1}) \prod_{j\geqslant n+1} (1 - S_j z)$$

and

$$[\mathcal{H}, \psi_n] = -\sum_m \psi_m A_{mn}$$

$$A_{mn} = \begin{cases} C_m S_{m-1} \cdots S_{n+1} C_n & \text{if} \quad m > n \\ 0 & \text{if} \quad m = n \\ (-1)^{n-m-1} C_n S_{n-1} \cdots S_{m+1} C_m & \text{if} \quad m < n\,. \end{cases} \tag{F.9}$$

We are to prove

$$\sum_{n\in\mathbb{Z}} A_{mn} x_n = (z + z^{-1}) x_m\,. \tag{F.10}$$

Suppose that $\beta_n = 0$, i.e. $C_n = 1$ and $S_n = 0$, except for $M \leqslant n \leqslant N$. If $m \geqslant N + 2$ or $m \leqslant M - 2$, then (F.10) is valid because

$$A_{mn} = \begin{cases} 1 & \text{if} \quad n = m \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_n = \begin{cases} z^n \prod_{M\leqslant j\leqslant N} (1 + S_j z^{-1}) & \text{if} \quad n \geqslant N + 1 \\ z^n \prod_{M\leqslant j\leqslant N} (1 - S_j z) & \text{if} \quad n \leqslant M - 1\,. \end{cases}$$

Therefore, (F.10) is written in the form

$$A^{(N,M)} x^{(N,M)} = 0 \tag{F.11}$$

where $A^{(N,M)} = \left(A^{(N,M)}_{mn}\right)_{N+1\geqslant m,n\geqslant M-1}$ and $x^{(N,M)} = \left(x^{(N,M)}_n\right)_{N+1\geqslant n\geqslant M-1}$. The matrix $A^{(N,M)}$ is of the form:

$$A^{(N,M)} = \left(\begin{array}{ccc} -z^{-1} & C_N & \cdots \\ C_N & -z - z^{-1} & \cdots \\ -S_N C_{N-1} & C_N C_{N-1} & \\ S_N S_{N-1} C_{N-2} & -C_N S_{N-1} C_{N-2} & \\ \cdot & \cdot & \\ \cdot & \cdot & \overline{A}^{(N-1,M)} \\ \cdot & \cdot & \\ (-1)^{N-M} S_N S_{N-1} \cdots S_{M+1} C_M & (-1)^{N-M-1} C_N S_{N-1} \cdots S_{M+1} C_M & \\ (-1)^{N-M+1} S_N S_{N-1} \cdots S_{M+1} S_M & (-1)^{N-M} C_N S_{N-1} \cdots S_{M+1} S_M & \end{array}\right)$$

$$= \left(\begin{array}{ccc} & S_N S_{N-1} \cdots S_{M+1} C_M & S_N S_{N-1} \cdots S_{M+1} S_M \\ & C_N S_{N-1} \cdots S_{M+1} C_M & C_N S_{N-1} \cdots S_{M+1} S_M \\ & \cdot & \cdot \\ \underline{A}^{(N,M+1)} & \cdot & \cdot \\ & \cdot & \cdot \\ & C_{M+2} S_{M+1} C_M & C_{M+2} S_{M+1} S_M \\ & C_{M+1} C_M & C_{M+1} S_M \\ \cdots & -z - z^{-1} & C_M \\ \cdots & C_M & -z \end{array}\right).$$

Here

$$\overline{A}^{(N-1,M)} = \left(A^{(N-1,M)}_{mn}\right)_{N-1\geqslant m,n\geqslant M-1} \qquad \underline{A}^{(N,M+1)} = \left(A^{(N,M+1)}_{mn}\right)_{N+1\geqslant m,n\geqslant M+1}.$$

Similarly, the vector $x^{(N,M)}$ is of the form

$$x^{(N,M)} = \begin{pmatrix} (1+S_M z^{-1})\cdots(1+S_N z^{-1})z^{N+1} \\ (1+S_M z^{-1})\cdots(1+S_{N-1}z^{-1})C_N z^N \\ \\ \overline{x}^{(N-1,M)}(1-S_N z) \end{pmatrix}$$

$$= \begin{pmatrix} \underline{x}^{(N,M+1)}(1+S_M z^{-1}) \\ \\ C_M(1-S_{M+1}z)\cdots(1-S_N z)z^M \\ (1-S_M z)(1-S_{M+1}z)\cdots(1-S_N z)z^{M-1} \end{pmatrix}. \tag{F.12}$$

Here

$$\overline{x}^{(N-1,M)} = \left(x^{(N-1,M)}_n\right)_{N-1\geqslant n\geqslant M-1} \qquad \underline{x}^{(N,M+1)} = \left(x^{(N,M+1)}_n\right)_{N+1\geqslant n\geqslant M+1}.$$

We prove (F.11) by induction on $N-M$. If $N=M$, we can check directly that

$$\begin{pmatrix} -z^{-1} & C_N & S_N \\ C_N & -z-z^{-1} & C_N \\ -S_N & C_N & -z \end{pmatrix}\begin{pmatrix} (1+S_N z^{-1})z^{N+1} \\ C_N z^N \\ (1-S_N z)z^{N-1} \end{pmatrix} = 0.$$

If $N>M$, noting that

$$-S_N\ (1+S_N z^{-1})z^{N+1} + C_N\ C_N z^N = (1-S_N z)z^N$$

we can reduce the equality $\left(A^{(N,M)}x^{(N,M)}\right)_n = 0$ for $N-1 \geqslant n \geqslant M-1$ to $\left(A^{(N-1,M)}x^{(N-1,M)}\right)_n = 0$. Similarly, noting that

$$C_M\ C_M z^M + S_M\ (1-S_M z)z^{M-1} = (1+S_M z^{-1})z^M$$

we can reduce the equality $\left(A^{(N,M)}x^{(N,M)}\right)_n = 0$ for $N+1 \geqslant n \geqslant M+1$ to $\left(A^{(N,M+1)}x^{(N,M+1)}\right)_n = 0$. The proof of (F.7) is similar.

Let us prove (F.8). We have

$$[\phi^*(\theta_1),\phi(\theta_2)]_+ = \sum_k C_k^2 z_2^k z_1^{-k} \prod_{j=-\infty}^{k-1}(1+S_j z_2^{-1})(1-S_j z_1)\prod_{j=k+1}^{\infty}(1-S_j z_2)(1+S_j z_1^{-1})$$

where $z_j = \mathrm{e}^{\mathrm{i}\theta_j}$ $(j=1,2)$. We assume that $C_n = 1$ and $S_n = 0$ except for $M \leqslant n \leqslant N$. Then we have

$$\begin{aligned}[\phi^*(\theta_1),\phi(\theta_2)]_+ &= z_2^{N+1}z_1^{-N-1}\frac{\prod_{j=M}^N(1+S_j z_2^{-1})(1-S_j z_1)}{1-z_2 z_1^{-1}} \\ &\quad + \sum_{k=M}^N z_2^k z_1^{-k}(1+S_k^2)\prod_{j=M}^{k-1}(1+S_j z_2^{-1})(1-S_j z_1)\prod_{j=k+1}^{N}(1-S_j z_2)(1+S_j z_1^{-1}) \\ &\quad + z_2^{M-1}z_1^{-M+1}\frac{\prod_{j=M}^N(1-S_j z_2)(1+S_j z_1^{-1})}{1-z_2^{-1}z_1}.\end{aligned} \tag{F.13}$$

Write the RHS as

$$\left(\sum_{k\in\mathbb{Z}} z_2^k z_1^{-k}\right)\prod_{j=M}^N(1+S_j z_1^{-1})(1-S_j z_1) + F^{(N,M)}(S_M,\ldots,S_N;z_1,z_2).$$

Then, $F^{(N,M)}(S_M,\ldots,S_N;z_1,z_2)$ belongs to $\mathbb{C}[S_M,\ldots,S_N,z_1,z_1^{-1},z_2,z_2^{-1}]$.

We wish to show $F^{(N,M)}=0$. Let $G^{(N,M)}(S_M,\ldots,S_N;z_1,z_2)$ be the RHS of (F.13). Note that $F^{(N,M)}=G^{(N,M)}$ in $\mathbb{C}(z_1,z_2)[S_M,\ldots,S_N]$. Therefore, it is enough to show $G^{(N,M)}=0$ as an element of $\mathbb{C}(z_1,z_2)[S_M,\ldots,S_N]$. Note that

$$\begin{aligned}G^{(N,M)}(S_M,\ldots,S_N;z_1,z_2)&=G^{(N,M)}(-S_M,\ldots,-S_N;z_2^{-1},z_1^{-1})\\&=z_2^{N+M}z_1^{-N-M}G^{(N,M)}(S_N,\ldots,S_M;z_2,z_1)\,.\end{aligned}$$

It is also easy to show that

$$G^{(N,M)}(S_M,\ldots,S_N;z_1,z_2)=G^{(N,M)}(S_{\sigma(M)},\ldots,S_{\sigma(N)};z_1,z_2)$$

for any permutation σ of $\{M,\ldots,N\}$. Now $G^{(N,M)}(S_M,\ldots,S_N;z_1,z_2)$ is a polynomial of degree 2 in S_M. Therefore, in order to show that $G^{(N,M)}=0$, it is enough to show

$$G^{(N,M)}(z_1,S_{M+1},\ldots,S_N;z_1,z_2)=0\,.$$

This is shown by induction: we have

$$G^{(N,M)}(z_1,S_{M+1},\ldots,S_N;z_1,z_2)=(z_2z_1^{-1}+z_2z_1)\overline{G}^{(N,M+1)}(S_{M+1},\ldots,S_N;z_1,z_2)$$

where

$$\begin{aligned}\overline{G}^{(N,M+1)}(S_{M+1},\ldots,S_N;z_1,z_2)&=-z_2^{N-1}z_1^{-N+1}\sum_{j=M+1}^{N}(1+S_jz_2^{-1})(1-S_jz_1)\\&\quad+(1-z_1z_2^{-1})\sum_{k=M+1}^{N}z_2^{k-1}z_1^{-k+1}(1+S_k^2)\\&\quad\times\prod_{j=M+1}^{k-1}(1+S_jz_2^{-1})(1-S_jz_1)\prod_{j=k+1}^{N}(1-S_jz_2)(1+S_jz_1^{-1})\\&\quad+\sum_{j=M+1}^{N}(1-S_jz_2)(1+S_jz_1^{-1})\,.\end{aligned}$$

Then we can show that

$$\begin{aligned}\overline{G}^{(N,M+1)}(S_{M+1},\ldots,S_N;z_1,z_2)&=z_2z_1^{-1}(1+S_{M+1}z_2^{-1})(1-S_{M+1}z_1)\\&\quad\times\overline{G}^{(N,M+2)}(S_{M+2},\ldots,S_N;z_1,z_2)\,.\end{aligned}$$

Because $\overline{G}^{(N,N)}(S_N;z_1,z_2)=0$, we have $\overline{G}^{(N,M+1)}(S_{M+1},\ldots,S_N;z_1,z_2)=0$ by induction.

F.4. Correlation functions

The vacuum vector $|\mathrm{vac}\rangle$ satisfies

$$\begin{aligned}&\phi(\theta)|\mathrm{vac}\rangle=0 &&\text{if}\quad -\pi/2\leqslant\theta\leqslant\pi/2\\&\phi^*(\theta)|\mathrm{vac}\rangle=0 &&\text{if}\quad \pi/2\leqslant\theta\leqslant 3\pi/2\,.\end{aligned}\tag{F.14}$$

Similarly, the dual vacuum $\langle\mathrm{vac}|$ satisfies

$$\begin{aligned}&\langle\mathrm{vac}|\phi(\theta)=0 &&\text{if}\quad \pi/2\leqslant\theta\leqslant 3\pi/2\\&\langle\mathrm{vac}|\phi^*(\theta)=0 &&\text{if}\quad -\pi/2\leqslant\theta\leqslant\pi/2\,.\end{aligned}\tag{F.15}$$

We also have $\langle\mathrm{vac}|\mathrm{vac}\rangle=1$. Our goal is to compute two point functions $\langle\mathrm{vac}|\psi_m^*\psi_n|\mathrm{vac}\rangle$. For this purpose we need

Proposition F.3.

$$\psi_n = \int_0^{2\pi} \phi(\theta) A_n(\theta) \frac{\mathrm{d}\theta}{2\pi} \qquad \psi_n^* = \int_0^{2\pi} \phi^*(\theta) A_n^*(\theta) \frac{\mathrm{d}\theta}{2\pi} \,.$$

Here $A_n(\theta)$ and $A_n^*(\theta)$ are given by

$$A_n(\theta) = \frac{\mathrm{e}^{-\mathrm{i}n\theta}}{C_n \prod_{j=-\infty}^{n-1}(1+S_j\,\mathrm{e}^{-\mathrm{i}\theta})\prod_{j=n+1}^{\infty}(1-S_j\,\mathrm{e}^{\mathrm{i}\theta})} \left\{ \frac{1}{1+S_n\,\mathrm{e}^{-\mathrm{i}\theta}} + \frac{1}{1-S_n\,\mathrm{e}^{\mathrm{i}\theta}} - 1 \right\}$$

$$A_n^*(\theta) = \frac{\mathrm{e}^{\mathrm{i}n\theta}}{C_n \prod_{j=-\infty}^{n-1}(1-S_j\,\mathrm{e}^{\mathrm{i}\theta})\prod_{j=n+1}^{\infty}(1+S_j\,\mathrm{e}^{-\mathrm{i}\theta})} \left\{ \frac{1}{1+S_n\,\mathrm{e}^{-\mathrm{i}\theta}} + \frac{1}{1-S_n\,\mathrm{e}^{\mathrm{i}\theta}} - 1 \right\}.$$

Proof. From equation (F.5) we have

$$\begin{aligned}
\int_0^{2\pi} \phi(\theta) A_n(\theta) \frac{\mathrm{d}\theta}{2\pi} &= \sum_{k\geqslant n} \psi_k \int_0^{2\pi} \frac{C_k \prod_{j=n}^{k-1}(1+S_j\,\mathrm{e}^{-\mathrm{i}\theta})}{C_n \prod_{j=n+1}^{k}(1-S_j\,\mathrm{e}^{\mathrm{i}\theta})} \\
&\quad\times \left\{ \frac{1}{1+S_n\,\mathrm{e}^{-\mathrm{i}\theta}} + \frac{1}{1-S_n\,\mathrm{e}^{\mathrm{i}\theta}} - 1 \right\} \mathrm{e}^{\mathrm{i}(k-n)\theta} \frac{\mathrm{d}\theta}{2\pi} \\
&\quad+ \sum_{k\leqslant n-1} \psi_k \int_0^{2\pi} \frac{C_k \prod_{j=k+1}^{n}(1-S_j\,\mathrm{e}^{\mathrm{i}\theta})}{C_n \prod_{j=k}^{n-1}(1+S_j\,\mathrm{e}^{-\mathrm{i}\theta})} \\
&\quad\times \left\{ \frac{1}{1+S_n\,\mathrm{e}^{-\mathrm{i}\theta}} + \frac{1}{1-S_n\,\mathrm{e}^{\mathrm{i}\theta}} - 1 \right\} \mathrm{e}^{-\mathrm{i}(n-k)\theta} \frac{\mathrm{d}\theta}{2\pi} \,.
\end{aligned}$$

Noting that $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta/2\pi = \delta_{n,0}$, we can show this is equal to ψ_n. The other case is similar.

The two point functions are given as follows. We have obviously

$$\langle \mathrm{vac}|\psi_m \psi_n|\mathrm{vac}\rangle = \langle \mathrm{vac}|\psi_m^* \psi_n^*|\mathrm{vac}\rangle = 0\,.$$

Proposition F.4. Suppose that $m < n$. We have

$$\begin{aligned}
\langle \mathrm{vac}|\psi_m^* \psi_n|\mathrm{vac}\rangle &= (-1)^{m-n+1} \langle \mathrm{vac}|\psi_n^* \psi_m|\mathrm{vac}\rangle \\
&= (-1)^{m-n} \langle \mathrm{vac}|\psi_m \psi_n^*|\mathrm{vac}\rangle = -\langle \mathrm{vac}|\psi_n \psi_m^*|\mathrm{vac}\rangle \\
&= \frac{\mathrm{i}^{m-n-1}}{\pi} (B_m B_n)^{1/2} \sum_{j=m}^{n} \beta_j \frac{\prod_{m+1\leqslant l\leqslant n-1}(B_j+B_l)}{\prod_{\substack{m\leqslant l\leqslant n \\ l\neq j}}(B_j-B_l)} \,.
\end{aligned} \tag{F.16}$$

Here, we set $B_j = \mathrm{e}^{\beta_j}$. In addition, we have

$$\langle \mathrm{vac}|\psi_n^* \psi_n|\mathrm{vac}\rangle = \langle \mathrm{vac}|\psi_n \psi_n^*|\mathrm{vac}\rangle = \tfrac{1}{2}\,.$$

Proof. Because of (F.2), it is enough to compute $\langle \mathrm{vac}|\psi_m^* \psi_n|\mathrm{vac}\rangle$ $(m, n \in \mathbb{Z})$. Consider the anti-involution

$$\psi_n \leftrightarrow \psi_n^* \qquad \beta_n \leftrightarrow -\beta_n \qquad \gamma_n \leftrightarrow -\gamma_n \qquad \langle \mathrm{vac}| \leftrightarrow |\mathrm{vac}\rangle \qquad \phi(\theta) \leftrightarrow \phi^*(-\theta)\,.$$

Note that the last expression in (F.16) changes sign by $(-1)^{m-n-1}$. Therefore, it is enough to prove the equality for $\langle \mathrm{vac}|\psi_m^* \psi_n|\mathrm{vac}\rangle$.

First, consider the case $m = n$. Using equations (F.14) and (F.15) and proposition F.3, we have

$$\begin{aligned}\langle \mathrm{vac}|\psi_n^*\psi_n|\mathrm{vac}\rangle &= \int_{\pi/2}^{3\pi/2} \frac{(1+S_n\,\mathrm{e}^{-\mathrm{i}\theta})(1-S_n\,\mathrm{e}^{\mathrm{i}\theta})}{C_n^2}\left\{\frac{1}{1+S_n\,\mathrm{e}^{-\mathrm{i}\theta}}+\frac{1}{1-S_n\,\mathrm{e}^{\mathrm{i}\theta}}-1\right\}^2\frac{\mathrm{d}\theta}{2\pi}\\ &= \int_{\pi/2}^{3\pi/2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\log\frac{1+S_n\,\mathrm{e}^{-\mathrm{i}\theta}}{1-S_n\,\mathrm{e}^{\mathrm{i}\theta}}+\mathrm{i}\theta\right)\frac{\mathrm{d}\theta}{2\pi\mathrm{i}}\\ &= \tfrac{1}{2}\,.\end{aligned}$$

In general, for $m < n$, we have

$$\langle \mathrm{vac}|\psi_m^*\psi_n|\mathrm{vac}\rangle = \int_{\pi/2}^{3\pi/2}\frac{C_m}{1+S_m\mathrm{e}^{-\mathrm{i}\theta}}\prod_{j=m+1}^{n-1}\frac{1-S_j\mathrm{e}^{\mathrm{i}\theta}}{1+S_j\mathrm{e}^{-\mathrm{i}\theta}}\frac{C_n}{1+S_n\mathrm{e}^{-\mathrm{i}\theta}}\mathrm{e}^{-\mathrm{i}(n-m)\theta}\frac{\mathrm{d}\theta}{2\pi}\,.$$

With the change of variable $z = -\mathrm{e}^{\mathrm{i}\theta}$, the right-hand side becomes

$$-\int_{-\mathrm{i}}^{\mathrm{i}}\frac{C_mC_n\prod_{j=m+1}^{n-1}(1+S_jz)}{\prod_{j=m}^{n}(-z+S_j)}\frac{\mathrm{d}z}{2\pi\mathrm{i}} = (-1)^{n-m}\frac{C_mC_n}{(2\pi\mathrm{i})^2}\int_C \mathrm{d}z\,\log\frac{z+\mathrm{i}}{z-\mathrm{i}}\frac{\prod_{j=m+1}^{n-1}(1+S_jz)}{\prod_{j=m}^{n}(z-S_j)}\,.$$

Here the branch of $\log(z+\mathrm{i})/(z-\mathrm{i})$ is such that it has the value 0 at $z=\infty$. The contour C is as in figure F1.

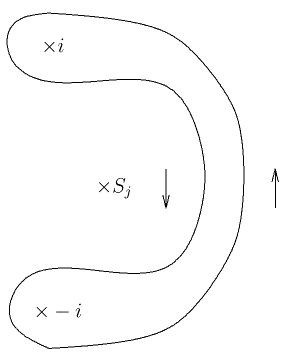


Figure F1. The contour C.

Taking the residues at $z = S_j$ $(m \leqslant j \leqslant n)$ and using (F.5) and, in particular, the equality

$$\log\frac{S_j+\mathrm{i}}{S_j-\mathrm{i}} = -\beta_j - \pi\mathrm{i} \qquad \sum_{j=m}^{n}\frac{\prod_{l=m+1}^{n-1}(B_j+B_l)}{\prod_{\substack{m\leqslant l\leqslant n\\ l\neq j}}(B_j-B_l)} = 0$$

we have (F.16).

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